United Kingdom
Mathematics Trust

Intermediate Mathematical Challenge

Follow-up Competitions

2003 – 2019 Collection

August 18, 2020

Comments and suggestions to 89272376@QQ.com .

IMC Follow-up 3

Year 9 Olympiad - Cayley

These problems are meant to be challenging! The earlier questions tend to be easier; later questions tend to be more demanding.

Do not hurry, but spend time working carefully on one question before attempting another.

Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **2 hours**.
3. The use of blank or lined paper for rough working, rulers and compasses is allowed; **squared paper, calculators and protractors are forbidden**.
4. You should write your solutions neatly on A4 paper. Staple your sheets together in the top left corner with the Cover Sheet on top and the questions in order.
5. Start each question on a fresh A4 sheet. **Do not hand in rough work**.
6. Your answers should be fully simplified, and exact. They may contain symbols such as π, fractions, or square roots, if appropriate, but not decimal approximations.
7. You should give full written solutions, including mathematical reasons as to why your method is correct. Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.

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1. Each of Alice and Beatrice has their birthday on the same day.

In 8 years' time, Alice will be twice as old as Beatrice. Ten years ago, the sum of their ages was 21.

How old is Alice now?

2. In the addition shown, each of the letters D, O, G, C, A and T represents a different digit.

What is the value of $D + O + G + C + A + T$?

$$\begin{array}{r} DOG \\ +\,CAT \\ \hline 1000 \end{array}$$

3. The triangle ABC is isosceles with $AB = BC$. The point D is a point on BC, between B and C, so that $AC = AD = BD$.

What is the size of angle ABC?

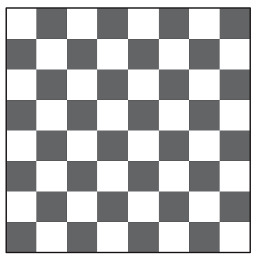
4. Arrange the digits 1, 2, 3, 4, 5, 6, 7, 8 to form two 4-digit integers whose difference is as small as possible.

Explain clearly why your arrangement achieves the smallest possible difference.

5. Howard chooses n different numbers from the list 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, so that no two of his choices add up to a square.

What is the largest possible value of n?

6. A chessboard is formed from an 8×8 grid of alternating black and white squares, as shown. The side of each small square is 1 cm.

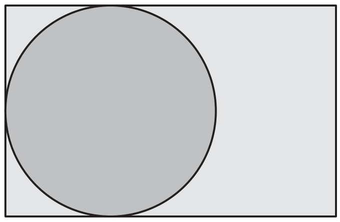


What is the largest possible radius of a circle that can be drawn on the board in such a way that the circumference is entirely on white squares or corners?

C1. The positive integer N has six digits in increasing order. For example, 124 689 is such a number.

However, unlike 124 689, three of the digits of N are 3, 4 and 5, and N is a multiple of 6.
How many possible six-digit integers N are there?

C2. A circle lies within a rectangle and touches three of its edges, as shown.



The area inside the circle equals the area inside the rectangle but outside the circle.

What is the ratio of the length of the rectangle to its width?

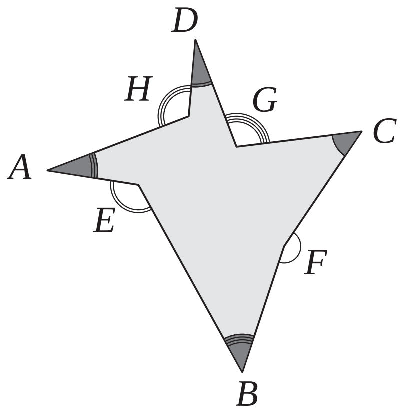
C3. The addition sum $XCV + XXV = CXX$ is true in Roman numerals.

In this question, however, the sum is actually the letter-sum shown alongside, in which: each letter stands for one of the digits 0 to 9, and stands for the same digit each time it occurs; different letters stand for different digits; and no number starts with a zero.

$$\begin{array}{r} XCV \\ +\,XXV \\ \hline CXX \end{array}$$

Find all solutions, and explain how you can be sure you have found every solution.

C4. Prove that the difference between the sum of the four marked interior angles A, B, C, D and the sum of the four marked exterior angles E, F, G, H of the polygon shown is 360°.

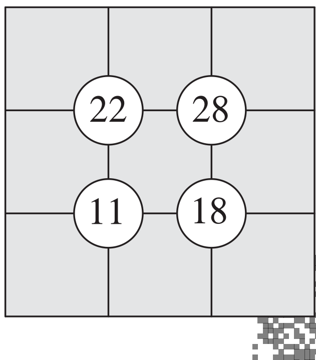


C5. In the expression below, three of the + signs are changed into − signs so that the expression is equal to 100:

$$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$
$$+ 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

In how many ways can this be done?

C6. In the puzzle *Suko*, the numbers from 1 to 9 are to be placed in the spaces (one number in each) so that the number in each circle is equal to the sum of the numbers in the four surrounding spaces.

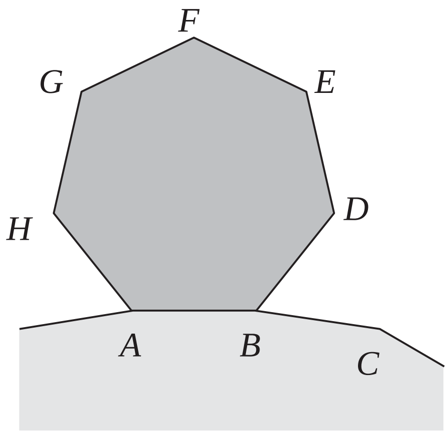


How many solutions are there to the *Suko* puzzle shown alongside?

C1. Four times the average of two different positive numbers is equal to three times the greater one. The difference between the numbers is three less than the average.

What are the two numbers?

C2.



The diagram shows three adjacent vertices *A*, *B* and *C* of a regular polygon with forty-two sides, and a regular heptagon *ABDEFGH*. The polygons are placed together edge-to-edge.

Prove that triangle *BCD* is equilateral.

C3. Peaches spends exactly £3.92 on some fruit, choosing from apples costing 20p each and pears costing 28p each.

How many of each type of fruit might she have bought?

C4. The point *X* lies inside the square *ABCD* and the point *Y* lies outside the square, in such a way that triangles *XAB* and *YAD* are both equilateral.

Prove that $XY = AC$.

C5. In a sports league there are four teams and every team plays every other team once. A team scores 3 points for a win, 1 point for a draw, and 0 points for a loss.

What is the smallest number of points that a team could have at the end of the league and still score more points than each of the other teams?

C6. We write '*pq*' to denote the two-digit integer with tens digit *p* and units digit *q*.

For which values of *a*, *b* and *c* are the two fractions $\frac{\text{`}ab\text{'}}{\text{`}ba\text{'}}$ and $\frac{\text{`}bc\text{'}}{\text{`}cb\text{'}}$ equal and different from 1?

C1. How many three-digit multiples of 9 consist only of odd digits?

C2. In a 6 × 6 grid of numbers:

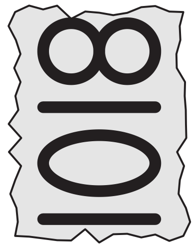
(i) all the numbers in the top row and the leftmost column are the same;

(ii) each other number is the sum of the number above it and the number to the left of it;

(iii) the number in the bottom right corner is 2016.

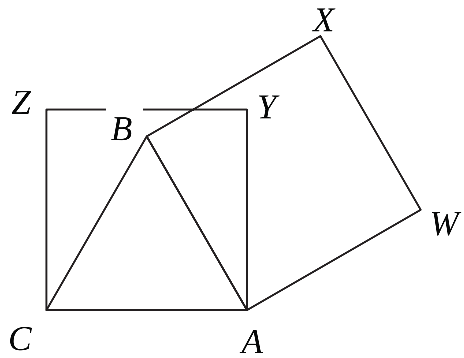
What are the possible numbers in the top left corner?

C3. All the telephone numbers in Georgetown have six digits and each of them begins with the digits 81. Kate finds the scrap of paper shown, with part of Jenny's telephone number on it.



How many different possibilities are there for Jenny's telephone number?

C4. The diagram shows an equilateral triangle *ABC* and two squares *AWXB* and *AYZC*.



Prove that triangle *AXZ* is equilateral.

C5. Dean wishes to place the positive integers 1, 2, 3, …, 9 in the cells of a 3 × 3 square grid so that:

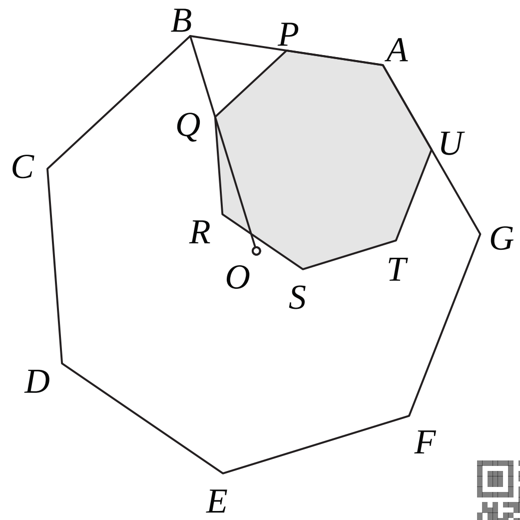
(i) there is exactly one number in each cell;

(ii) the product of the numbers in each row is a multiple of four;

(iii) the product of the numbers in each column is a multiple of four.

Is Dean's task possible? Prove that your answer is correct.

C6. The diagram shows two regular heptagons *ABCDEFG* and *APQRSTU*. The vertex *P* lies on the side *AB* (and hence *U* lies on the side *GA*). Also, *Q* lies on *OB*, where *O* is the centre of the larger heptagon.



Prove that $AB = 2AP$.

1. A train travelling at constant speed takes five seconds to pass completely through a tunnel which is 85 m long, and eight seconds to pass completely through a second tunnel which is 160 m long.

What is the speed of the train?

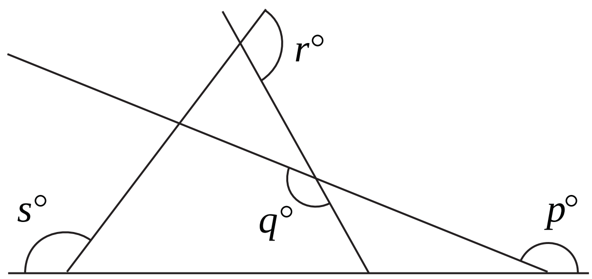
2. The integers a, b, c, d, e, f and g, none of which is negative, satisfy the following five simultaneous equations:

$$\begin{aligned} a + b + c &= 2 \\ b + c + d &= 2 \\ c + d + e &= 2 \\ d + e + f &= 2 \\ e + f + g &= 2. \end{aligned}$$

What is the maximum possible value of $a + b + c + d + e + f + g$?

3. Four straight lines intersect as shown.

What is the value of $p + q + r + s$?

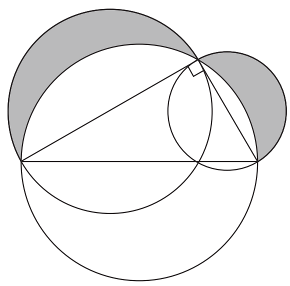


4. Ten balls, each coloured green, red or blue, are placed in a bag. Ten more balls, each coloured green, red or blue, are placed in a second bag.
In one of the bags there are at least seven blue balls and in the other bag there are at least four red balls. Overall there are half as many green balls as there are blue balls.

Prove that the total number of red balls in both bags is equal to either the total number of blue balls in both bags or the total number of green balls in both bags.

5. The diagram shows a right-angled triangle and three circles. Each side of the triangle is a diameter of one of the circles. The shaded region R is the region inside the two smaller circles but outside the largest circle.

Prove that the area of R is equal to the area of the triangle.



6. I have four identical black beads and four identical white beads.

Carefully explain how many different bracelets I can make using all the beads.

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