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Sixth Term Examination Paper

23-S3

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Section A: Pure Mathematics

1 The distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the curve $x_2 = 4ay$, where $a > 0$.

(i) Given that

$$(p+q)^2 = pq^2 + 6pq + 5, \qquad (\star)$$

show that the line through P and Q is a tangent to the circle with centre $(0, 3a)$ and radius $2a$.

(ii) Show that, for any given value of p with $p^2 \neq 1$, there are two distinct real values of q that satisfy equation $(\star)$.

Let these values be q_1 and q_2. Find expressions, in terms of p, for $q_1 + q_2$ and $q_1 q_2$.

(iii) Show that, for any given value of p with $p^2 \neq 1$, there is a triangle with one vertex at P such that all three vertices lie on the curve $x_2 = 4ay$ and all three sides are tangents to the circle with centre $(0, 3a)$ and radius $2a$.

2 The polar curves C_1 and C_2 are defined for $0 \leqslant \theta \leqslant \pi$ by

$$r = k(1 + \sin\theta)$$
$$r = k + \cos\theta$$

respectively, where k is a constant greater than 1.

(i) Sketch the curves on the same diagram. Show that if $\theta = \alpha$ at the point where the curves intersect, $\tan\alpha = \frac{1}{k}$.

(ii) The region A is defined by the inequalities

$$0 \leqslant \theta \leqslant \alpha \quad \text{and} \quad r \leqslant k(1 + \sin\theta).$$

Show that the area of A can be written as

$$\frac{k^2}{4}(3\alpha - \sin\alpha\cos\alpha) + k^2(1 - \cos\alpha).$$

(iii) The region B is defined by the inequalities

$$\alpha \leqslant \theta \leqslant \pi \quad \text{and} \quad r \leqslant k + \cos\theta.$$

Find an expression in terms of k and α for the area of B.

(iv) The total area of regions A and B is denoted by R.
The area of the region enclosed by C_1 and the lines $\theta = 0$ and $\theta = \pi$ is denoted by S.
The area of the region enclosed by C_2 and the lines $\theta = 0$ and $\theta = \pi$ is denoted by T.

Show that as $k \to \infty$,

$$\frac{R}{T} \to 1$$

and find the limit of

$$\frac{R}{S}$$

as $k \to \infty$.

3 **(i)** Show that, if a and b are complex numbers, with $b \neq 0$, and s is a positive real number, then the points in the Argand diagram representing the complex numbers $a + sb\mathbf{i}$, $a - sb\mathbf{i}$ and $a + b$ form an isosceles triangle.

Given three points which form an isosceles triangle in the Argand diagram, explain with the aid of a diagram how to determine the values of a, b and s, so that the vertices of the triangle represent complex numbers $a + sb\mathbf{i}, a - sb\mathbf{i}$ and $a + b$.

(ii) Show that, if the roots of the equation $z^3 + pz + q = 0$, where p and q are complex numbers, are represented in the Argand diagram by the vertices of an isosceles triangle, then there is a non-zero real number s such that

$$\frac{p^3}{q^2} = \frac{27(3s^2 - 1)^3}{4(9s^2 + 1)^2}.$$

(iii) Sketch the graph $y = \dfrac{(3x-1)^3}{(9x+1)^2}$, identifying any stationary points.

(iv) Show that if the roots of the equation $z^3 + pz + q = 0$ are represented in the Argand diagram by the vertices of an isosceles triangle then $\dfrac{p^3}{q^2}$ is a real number and $\dfrac{p^3}{q^2} > -\dfrac{27}{4}$.

4 Let n be a positive integer. The polynomial p is defined by the identity

$$\mathrm{p}(\cos\theta) \equiv \cos\left((2n+1)\theta\right) + 1.$$

(i) Show that

$$\cos\left((2n+1)\theta\right) = \sum_{r=0}^{n} \binom{2n+1}{2r} \cos^{2n+1-2r}(\theta) \left(\cos^2\theta\right)^r.$$

(ii) By considering the expansion of $(1+t)^{2n+1}$ for suitable values of t, show that the coefficient of x^{2n+1} in the polynomial $\mathrm{p}(x)$ is 2^{2n}.

(iii) Show that the coefficient of x^{2n-1} in the polynomial $\mathrm{p}(x)$ is $-(2n+1)2^{2n-2}$.

(iv) It is given that there exists a polynomial q such that

$$\mathrm{p}(x) = (x+1)[\mathrm{q}(x)]^2$$

and the coefficient of x^n in $\mathrm{q}(x)$ is greater than 0.

Write down the coefficient of x^n in the polynomial $\mathrm{q}(x)$ and, for $n \geqslant 2$, show that the coefficient of x^{n-2} in the polynomial $\mathrm{q}(x)$ is

$$2^{n-2}(1-n).$$

5 **(i)** Show that if

$$\frac{1}{x} + \frac{2}{y} = \frac{2}{7},$$

then $(2x - 7)(y - 7) = 49$.

By considering the factors of 49, find all the pairs of positive integers x and y such that

$$\frac{1}{x} + \frac{2}{y} = \frac{2}{7}.$$

(ii) Let p and q be prime numbers such that

$$p^2 + pq + q^2 = n^2$$

where n is a positive integer. Show that

$$(p + q + n)(p + q.n) = pq$$

and hence explain why $p + q = n + 1$.

Hence find the possible values of p and q.

(iii) Let p and q be positive and

$$p^3 + q^3 + 3pq^2 = n^3.$$

Show that $p + q - n < p$ and $p + q - n < q$.

Show that there are no prime numbers p and q such that $p^3 + q^3 + 3pq^2$ is the cube of an integer.

6 **(i)** By considering the Maclaurin series for e^x, show that for all real x,

$$\cosh^2 x \geqslant 1 + x^2.$$

Hence show that the function f, defined for all real x by $\mathrm{f}(x) = \tan^{-1} x - \tanh x$, is an increasing function.

Sketch the graph $y = \mathrm{f}(x)$.

(ii) Function g is defined for all real x by $\mathrm{g}(x) = \tan^{-1} x - \frac{1}{2}\pi \tanh x$.

(a) Show that g has at least two stationary points.

(b) Show, by considering its derivative, that $(1 + x^2)\sinh x - x\cosh x$ is non-negative for $x \geqslant 0$.

(c) Show that $\dfrac{\cosh^2 x}{1 + x^2}$ is an increasing function for $x \geqslant 0$.

(d) Hence or otherwise show that g has exactly two stationary points.

(e) Sketch the graph $y = \mathrm{g}(x)$.

7 **(i)** Let f be a continuous function defined for $0 \leqslant x \leqslant 1$. Show that

$$\int_0^1 \mathrm{f}\left(\sqrt{x}\right)\,\mathrm{d}x = 2\int_0^1 x\mathrm{f}(x)\,\mathrm{d}x\,.$$

(ii) Let g be a continuous function defined for $0 \leqslant x \leqslant 1$ such that

$$\int_0^1 \left(\mathrm{g}(x)\right)^2\,\mathrm{d}x = \int_0^1 \mathrm{g}(\sqrt{x})\,\mathrm{d}x - \frac{1}{3}.$$

Show that $\displaystyle\int_0^1 \left(\mathrm{g}(x) - x\right)^2\,\mathrm{d}x = 0$ and explain why $\mathrm{g}(x) = x$ for $0 \leqslant x \leqslant 1$.

(iii) Let h be a continuous function defined for $0 \leqslant x \leqslant 1$ with derivative h' such that

$$\int_0^1 \left(\mathrm{h}'(x)\right)^2\,\mathrm{d}x = 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\,\mathrm{d}x - \frac{1}{3}.$$

Given that $\mathrm{h}(0) = 0$, find h.

(iv) Let k be a continuous function defined for $0 \leqslant x \leqslant 1$ and a be a real number, such that

$$\int_0^1 \mathrm{e}^{ax}\left(\mathrm{k}(x)\right)^2\,\mathrm{d}x = 2\int_0^1 \mathrm{k}(x)\,\mathrm{d}x + \frac{\mathrm{e}^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4}.$$

Show that a must be equal to 2 and find k.

8 If

$$y = \begin{cases} \mathrm{k}_1(x) & x \leqslant b \\ \mathrm{k}_2(x) & x \geqslant b \end{cases}$$

with $\mathrm{k}_1(b) = \mathrm{k}_2(b)$, then y is said to be *continuously diffierentiable* at $x = b$ if $\mathrm{k}_1'(b) = \mathrm{k}_2'(b)$.

(i) Let $\mathrm{f}(x) = x\mathrm{e}^{-x}$. Verify that, for all real x, $y = \mathrm{f}(x)$ is a solution to the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

and that $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Show that $\mathrm{f}'(x) \geqslant 0$ for $x \leqslant 1$.

(ii) You are given the diffierential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + y = 0$$

where $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$. Let

$$y = \begin{cases} \mathrm{g}_1(x) & x \leqslant 1 \\ \mathrm{g}_2(x) & x \geqslant 1 \end{cases}$$

be a solution of the differential equation which is continuously differentiable at $x = 1$.

Write down an expression for $\mathrm{g}_1(x)$ and find an expression for $\mathrm{g}_2(x)$.

(iii) State the geometrical relationship between the curves $y = \mathrm{g}_1(x)$ and $y = \mathrm{g}_2(x)$.

(iv) Prove that if $y = k(x)$ is a solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p\frac{\mathrm{d}y}{\mathrm{d}x} + qy = 0$$

in the interval $r \leqslant x \leqslant s$, where p and q are constants, then, in a suitable interval which you should state, $y = k(c - x)$ satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - p\frac{\mathrm{d}y}{\mathrm{d}x} + qy = 0.$$

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(v) You are given the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + 2y = 0$$

where $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Let $\mathrm{h}(x) = \mathrm{e}^{-x}\sin x$. Show that $\mathrm{h}'\left(\frac{1}{4}\pi\right) = 0$.

It is given that $y = \mathrm{h}(x)$ satisfies the differential equation in the interval $-\frac{3}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$. and that $\mathrm{h}'(x) \geqslant 0$ in this interval.

In a solution to the differential equation which is continuously differentiable at $\left(n + \frac{1}{4}\right)\pi$ for all $n \in \mathbb{Z}$, find y in terms of x in the intervals

(a) $\frac{1}{4}\pi \leqslant x \leqslant \frac{5}{4}\pi$,

(b) $\frac{5}{4}\pi \leqslant x \leqslant \frac{9}{4}\pi$.

Section B: Mechanics

9 Two particles, A of mass m and B of mass M, are fixed to the ends of a light inextensible string AB of length r and lie on a smooth horizontal plane. The origin of coordinates and the x–and y–axes are in the plane.

Initially, A is at $(0, 0)$ and B is $(r, 0)$. B is at rest and A is given an instantaneous velocity of magnitude u in the positive y direction.

At a time t after this, A has position (x, y) and B has position (X, Y). You may assume that, in the subsequent motion, the string remains taut.

(i) Explain by means of a diagram why

$$X = x + r\cos\theta$$
$$Y = y - r\sin\theta$$

where θ is the angle *clockwise* from the positive x-axis of the vector $\overrightarrow{AB}$.

(ii) Find expressions for $\dot{X}, \dot{Y}, \ddot{X}$ and $\ddot{Y}$ in terms of $\ddot{x}, \ddot{y}, \dot{x}, \dot{y}, r, \ddot{\theta}, \dot{\theta}$ and θ, as appropriate.

Assume that the tension T in the string is the only force acting on either particle.

(iii) Show that

$$\ddot{x}\sin\theta + \ddot{y}\cos\theta = 0$$
$$\ddot{X}\sin\theta + \ddot{Y}\cos\theta = 0$$

and hence that $\theta = \dfrac{ut}{r}$.

(iv) Show that

$$m\ddot{x} + M\ddot{X} = 0$$
$$m\ddot{y} + M\ddot{Y} = 0$$

and find $my + MY$ in terms of t and m, M, u, r as appropriate.

(v) Show that

$$y = \frac{1}{m+M}\left(mut + Mr\sin\left(\frac{ut}{r}\right)\right).$$

(vi) Show that, if $M > m$, then the y component of the velocity of particle A will be negative at some time in the subsequent motion.

10 A thin uniform beam AB has mass $3m$ and length $2h$. End A rests on rough horizontal ground and the beam makes an angle of 2β to the vertical, supported by a light inextensible string attached to end B. The coefficient of friction between the beam and the ground at A is μ.

The string passes over a small frictionless pulley fixed to a point C which is a distance $2h$ vertically above A. A particle of mass km, where $k < 3$, is attached to the other end of the string and hangs freely.

(i) Given that the system is in equilibrium, find an expression for k in terms of β and show that $k^2 \leqslant \dfrac{9\mu^2}{\mu^2+1}$.

(ii) A particle of mass m is now fixed to the beam at a distance xh from A, where $0 \leqslant x \leqslant 2$. Given that $k = 2$, and that the system is in equilibrium, show that

$$\frac{F^2}{N^2} = \frac{x^2+6x+5}{4(x+2)^2},$$

where F is the frictional force and N is the normal reaction on the beam at A.

By considering $\dfrac{1}{3} - \dfrac{F^2}{N^2}$, or otherwise, find the minimum value of μ for which the beam can be in equilibrium whatever the value of x.

Section C: Probability and Statistics

11 Show that

$$\sum_{k=1}^{\infty} \frac{k+1}{k!} x^k = (x+1)\mathrm{e}^x - 1.$$

In the remainder of this question, n is a fixed positive integer.

(i) Random variable Y has a Poisson distribution with mean n. One observation of Y is taken. Random variable D is defined as follows. If the observed value of Y is zero then $D = 0$. If the observed value of Y is k, where $k \geqslant 1$, then a fair k-sided die (with sides numbered 1 to k) is rolled once and D is the number shown on the die.

(a) Write down $\mathrm{Pr}(D = 0)$.

(b) Show, from the definition of the expectation of a random variable, that

$$\mathrm{E}(D) = \sum_{d=1}^{\infty} \left[d \sum_{k=d}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \mathrm{e}^{-n} \right) \right].$$

Show further that

$$\mathrm{E}(D) = \sum_{k=1}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \mathrm{e}^{-n} \sum_{d=1}^{k} d \right).$$

(c) Show that $\mathrm{E}(D) = \frac{1}{2}(n + 1 - \mathrm{e}^{-n})$.

(ii) Random variables $X_1, X_2, \ldots, X_n$ all have Poisson distributions.
For each $k \in 1, 2, \ldots, n$, the mean of X_k is k.

A fair n-sided die, with sides numbered 1 to n, is rolled. When k is the number shown, one observation of X_k is recorded. Let Z be the number recorded.

(a) Find $\mathrm{Pr}(Z = 0)$.

(b) Show that $\mathrm{E}(Z) > \mathrm{E}(D)$.

12 A drawer contains n pairs of socks. The two socks in each pair are indistinguishable, but each pair of socks is a different colour from all the others. A set of $2k$ socks, where k is an integer with $2k \leqslant n$, is selected at random from this drawer: that is, every possible set of $2k$ socks is equally likely to be selected.

(i) Find the probability that, among the socks selected, there is no pair of socks.

(ii) Let $X_{n,k}$ be the random variable whose value is the number of pairs of socks found amongst those selected. Show that

$$\Pr(X_{n,k} = r) = \frac{\dbinom{n}{r}\dbinom{n-r}{2(k-r)}2^{2(k-r)}}{\dbinom{2n}{2k}}$$

for $0 \leqslant r \leqslant k$.

(iii) Show that

$$r\,\Pr(X_{n,k} = r) = \frac{k(2k-1)}{2n-1}\,\Pr(X_{n-1,k-1} = r-1),$$

for $1 \leqslant r \leqslant k$, and hence find $\mathrm{E}(X_{n,k})$.