There are 12 questions in this paper.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12

Sixth Term Examination Paper

22-S3

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Section A: Pure Mathematics

1 Let C_1 be the curve given by the parametric equations

$$x = ct \quad \text{and} \quad y = \frac{c}{t},$$

where $c > 0$ and $t \neq 0$, and let C_2 be the circle

$$(x-a)^2 + (y-b)^2 = r^2.$$

C_1 and C_2 intersect at the four points $P_i\,(i = 1, 2, 3, 4)$, and the corresponding values of the parameter t at these points are t_i.

(i) Show that t_i are the roots of the equation

$$c^2t^4 - 2act^3 + (a^2 + b^2 - r^2)t^2 - 2bct + c^2 = 0. \qquad (*)$$

(ii) Show that

$$\sum_{i=1}^{4} t_i^2 = \frac{2}{c^2}\left(a^2 - b^2 + r^2\right)$$

and find a similar expression for $\displaystyle\sum_{i=1}^{4} \frac{1}{t_i^2}$.

(iii) Hence show that $\displaystyle\sum_{i=1}^{4} OP_i^2 = 4r^2$, where OP_i denotes the distance of the point P_i from the origin.

(iv) Suppose that the curves C_1 and C_2 touch at two distinct points.

By considering the product of the roots of $(*)$, or otherwise, show that the centre of circle C_2 must lie on either the line $y = x$ or $y = -x$.

2 **(i)** Suppose that there are three non-zero integers a, b and c for which $a^3 + 2b^3 + 4c^3 = 0$.

Explain why there must exist an integer p, with $|p| < |a|$, such that $4p^3 + b^3 + 2c^3 = 0$, and show further that there must exist integers p, q and r, with $|p| < |a|, |q| < |b|$ and $3|r| < |c|$, such that $p^3 + 2q^3 + 4r = 0$.

Deduce that no such integersa, b and c can exist.

(ii) Prove that there are no non-zero integers a, b and c for which $9a^3 + 10b^3 + 6c^3 = 0$.

(iii) By considering the expression $(3n \pm 1)^2$, prove that, unless an integer is a multiple of three, its square is one more than a multiple of 3.

Deduce that the sum of the squares of two integers can only be a multiple of three if each of the integers is a multiple of three.

Hence prove that there are no non-zero integers a, b and c for which $a^2 + b^2 = 3c$.

(iv) Prove that there are no non-zero integers a, b and c for which $a^2 + b^2 + c^2 = 4abc$.

3 **(i)** The curve C_1 has equation

$$ax^2 + bxy + cy^2 = 1$$

where $abc \neq 0$ and $a > 0$.

Show that, if the curve has two stationary points, then $b^2 < 4ac$.

(ii) The curve C_2 has equation

$$ay^3 + bx^2y + cx = 1$$

where $abc \neq 0$ and $b > 0$.

Show that the x-coordinates of stationary points on this curve satisfy

$$4cb^3x^4 - 8b^3x^3 - ac^3 = 0.$$

Show that, if the curve has two stationary points, then $4ac^6 + 27b^3 > 0$.

(iii) Consider the simultaneous equations

$$\begin{aligned} ay^3 + bx^2y + cx &= 1 \\ 2bxy + c &= 0 \\ 3ay^2 + bx^2 &= 0 \end{aligned}$$

where $abc \neq 0$ and $b > 0$.

Show that, if these simultaneous equations have a solution, then $4ac^6 + 27b^3 = 0$.

4 *You may assume that all infinite sums and products in this question converge.*

(i) Prove by induction that for all positive integers n,

$$\sinh(x) = 2^n \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\sinh\left(\frac{x}{2^n}\right)$$

and deduce that, for $x \neq 0$,

$$\frac{\sinh(x)}{x}\cdot\frac{\frac{x}{2^n}}{\sinh\left(\frac{x}{2^n}\right)} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right).$$

(ii) You are given that the Maclaurin series for $\sinh x$ is

$$\sinh x = \sum_{r=0}^{\infty}\frac{x^{2r+1}}{(2r+1)!}.$$

Use this result to show that, as y tends to 0, $\dfrac{y}{\sinh y}$ tends to 1.
Deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\cdots.$$

(iii) Let $x = \ln 2$. Evaluate $\cosh\left(\dfrac{x}{2}\right)$ and show that

$$\cosh\left(\frac{x}{4}\right) = \frac{1+2^{\frac{1}{2}}}{2\times 2^{\frac{1}{4}}}.$$

Use part **(ii)** to show that

$$\frac{1}{\ln 2} = \frac{1+2^{\frac{1}{2}}}{2}\times\frac{1+2^{\frac{1}{4}}}{2}\times\frac{1+2^{\frac{1}{8}}}{2}\cdots.$$

(iv) Show that

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\times\frac{\sqrt{2+\sqrt{2}}}{2}\times\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots.$$

5 **(i)** Show that

$$\int_{-a}^{a} \frac{1}{1+\mathrm{e}^x}\,\mathrm{d}x = a$$

for all $a \geqslant 0$.

(ii) Explain why, if g is a continuous function and

$$\int_{0}^{a} \mathrm{g}(x)\,\mathrm{d}x = 0$$

for all $a \geqslant 0$, then $\mathrm{g}(x) = 0$ for all $x \geqslant 0$.

Let f be a continuous function with $\mathrm{f}(x) \geqslant 0$ for all x. Show that

$$\int_{-a}^{a} \frac{1}{1+\mathrm{f}(x)}\,\mathrm{d}x = a$$

for all $a \geqslant 0$, if and only if

$$\frac{1}{1+\mathrm{f}(x)} + \frac{1}{1+\mathrm{f}(-x)} - 1 = 0$$

for all $x \geqslant 0$, and hence if and only if $\mathrm{f}(x)\mathrm{f}(-x) = 1$ for all x.

(iii) Let f be a continuous function such that, for all $x, \mathrm{f}(x) \geqslant 0$ and $\mathrm{f}(x)\mathrm{f}(-x) = 1$.

Show that,if h is a continuous function with $\mathrm{h}(x) = \mathrm{h}(-x)$ for all x, then

$$\int_{-a}^{a} \frac{\mathrm{h}(x)}{1+\mathrm{f}(x)}\,\mathrm{d}x = \int_{0}^{a} \mathrm{h}(x)\,\mathrm{d}x\,.$$

(iv) Hence find the exact value of

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\mathrm{e}^{-x}\cos(x)}{\cosh(x)}\,\mathrm{d}x\,.$$

6 **(i)** Show that when α is small,

$$\cos(\theta + \alpha) - \cos\theta \approx -\alpha\sin\theta - \frac{1}{2}\alpha^2\cos\theta.$$

Find the limit as $\alpha \to 0$ of

$$\frac{\sin(\theta + \alpha) - \sin\theta}{\cos(\theta + \alpha) - \cos\theta} \qquad (\star)$$

in the case $\sin\theta \neq 0$.

In the case $\sin\theta = 0$, what happens to the value of expression $(\star)$ when $\alpha \to 0$?

(ii) A circle C_1 of radius a rolls without slipping in an anti-clockwise direction on a fixed circle C_2 with centre at the origin O and radius $(n-1)a$, where n is an integer greater than 2. The point P is fixed on C_1. Initially the centre of C_1 is at $(na, 0)$ and P is at $((n+1)a, 0)$.

(a) Let Q be the point of contact of C_1 and C_2 at any time in the rolling motion. Show that when OQ makes an angle θ, measured anticlockwise, with the positive x-axis, the x-coordinate of P is $x(\theta) = a(n\cos\theta + \cos n\theta)$, and find the corresponding expression for the y-coordinate, $y(\theta)$, of P.

(b) Find the values of θ for which the distance OP is $(n-1)a$.

(c) Let $\theta_0 = \dfrac{1}{n-1}\pi$. Find the limit as $\alpha \to 0$ of

$$\frac{y(\theta_0 + \alpha) - y(\theta_0)}{x(\theta_0 + \alpha) - x(\theta_0)}.$$

Hence show that, at the point $(x(\theta_0), y(\theta_0))$, the tangent to the curve traced out by P is parallel to OP.

7 Let $\mathbf{n}$ be a vector of unit length in three dimensions. For each vector $\mathbf{r}$, $\mathrm{f}(\mathbf{r})$ is defined by $\mathrm{f}(\mathbf{r}) = \mathbf{n} \times \mathbf{r}$.

(i) Given that

$$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

show that the x-component of $\mathrm{f}(\mathrm{f}(\mathbf{r}))$ is $-x(b^2 + c^2) + aby + acz$.

Show further that

$$\mathrm{f}(\mathrm{f}(\mathbf{r})) = (\mathbf{n}.\mathbf{r})\mathbf{n} - \mathbf{r}.$$

Explain, by means of a diagram, how $\mathrm{f}(\mathrm{f}(\mathbf{r}))$ is related to $\mathbf{n}$ and $\mathbf{r}$.

(ii) Let R be the point with position vector $\mathbf{r}$ and P be the point with position vector $\mathrm{g}(\mathbf{r})$, where g is defined by

$$\mathrm{g}(s) = \mathbf{s} + \sin\theta \mathrm{f}(\mathbf{s}) + (1 - \cos\theta)\mathrm{f}(\mathrm{f}(\mathbf{s})).$$

By considering $\mathrm{g}(\mathbf{n})$ and $\mathrm{g}(\mathbf{r})$ when $\mathbf{r}$ is perpendicular to $\mathbf{n}$, state, with justification, the geometric transformation which maps R onto P.

(iii) Let R be the point with position vector $\mathbf{r}$ and Q be the point with position vector $\mathrm{h}(\mathbf{r})$, where h is defined by

$$\mathrm{h}(\mathbf{s}) = -\mathbf{s} - 2\mathrm{f}(\mathrm{f}(\mathbf{s})).$$

State, with justification, the geometric transformation which maps R onto Q.

8 **(i)** Use De Moivre's theorem to prove that for any positive integer $k > 1$,

$$\sin(k\theta) = \sin\theta\cos^{k-1}\theta\left(k - \binom{k}{3}\left(\sec^2\theta - 1\right) + \binom{k}{5}\left(\sec^2\theta - 1\right)^2 - \cdots\right)$$

and find a similar expression for $\cos(k\theta)$.

(ii) Let $\theta = \cos^{-1}(\frac{1}{a})$, where θ is measured in degrees, and a is an odd integer greatera than 1.

Suppose that there is a positive integer k such that $\sin(k\theta) = 0$ and $\sin(m\theta) \neq 0$ for all integers m with $0 < m < k$.

Show that it would be necessary to have k even and $\cos\left(\frac{1}{2}k\theta\right) = 0$.

Deduce that θ is irrational.

(iii) Show that if $\phi = \cot^{-1}\left(\frac{1}{b}\right)$, where ϕ is measured in degrees, and b is an even integer greater than 1, then ϕ is irrational.

Section B: Mechanics

9 **(i)** Two particles A and B, of masses m and km respectively, lie at rest on a smooth horizontal surface. The coefficient of restitution between the particles is e, where $0 < e < 1$. Particle A is then projected directly towards particle B with speed u.

Let v_1 and v_2 be the velocities of particles A and B, respectively, after the collision, in the direction of the initial velocity of A.

Show that $v_1 = \alpha u$ and $v_2 = \beta u$, where $\alpha = \dfrac{1-ke}{k+1}$ and $\beta = \dfrac{1+e}{k+1}$.

Particle B strikes a vertical wall which is perpendicular to its direction of motion and a distance D from the point of collision with A, and rebounds. The coefficient of restitution between particle B and the wall is also e.

Show that, if A and B collide for a second time at a point D from the wall, then $k = \dfrac{1+e-e^2}{e(2e+1)}$.

(ii) Three particles A, B and C, of masses m, km and k^2m respectively, lie at rest on a smooth horizontal surface in a straight line, with B between A and C. A vertical wall is perpendicular to this line and lies on the side of C away from A and B. The distance between B and C is equal to d and the distance between C and the wall is equal to $3d$. The coefficient of restitution between each pair of particles, and between particle C and the wall, is e, where $0 < e < 1$. Particle A is then projected directly towards particle B with speed u.

Show that, if all three particles collide simultaneously at a point $\frac{3}{2}d$ from the wall, then $e = \frac{1}{2}$.

10 Two light elastic springs each have natural length a. One end of each spring is attached to a particle P of weight W. The other ends of the springs are attached to the end-points, B and C, of a fixed horizontal bar BC of length $2a$. The moduli of elasticity of the springs PB and PC are s_1W and s_2W respectively; these values are such that the particle P hangs in equilibrium with angle BPC equal to $90°$.

(i) Let angle $PBC = \theta$. Show that $s_1 = \dfrac{\sin\theta}{2\cos(\theta)-1}$ and find s_2 in terms of θ.

(ii) Take the zero level of gravitational potential energy to be the horizontal bar BC and let the total potential energy of the system be $-paW$. Show that p satisfies

$$\frac{1}{2}\sqrt{2} \geqslant p > \frac{1}{4}(1+\sqrt{3})$$

and hence that $p = 0.7$, correct to one significant figure.

Section C: Probability and Statistics

11 A fair coin is tossed N times and the random variable X records the number of heads. The mean deviation, δ, of X is defined by

$$\delta = E|X - \mu|,$$

where μ is the mean of X.

(i) Let $N = 2n$ where n is a positive integer.

(a) Show that $\Pr(X \leqslant n-1) = \frac{1}{2}\left(1 - \Pr(X = n)\right)$.

(b) Show that

$$\delta = \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\frac{1}{2^{2n-1}}.$$

(c) Show that for $r > 0$,

$$r\binom{2n}{r} = 2n\binom{2n-1}{r-1}.$$

Hence show that

$$\delta = \frac{n}{2^{2n}}\binom{2n}{n}.$$

(ii) Find a similar expression for δ in the case $N = 2n+1$.

12 **(i)** The point A lies on the circumference of a circle of radius a and centre O. The point B is chosen at random on the circumference, so that the angle AOB has a uniform distribution on $[0, 2\pi]$. Find the expected length of the chord AB.

(ii) The point C is chosen at random in the interior of a circle of radius a and centre O, so that the probability that it lies in any given region is proportional to the area of the region. The random variable R is defined as the distance between C and O.

Find the probability density function of R.

Obtain a formula in terms of a, R and t for the length of a chord through C that makes an acute angle of t with OC.

Show that as C varies (with t fixed), the expected length $L(t)$ of such chords is given by

$$L(t) = \frac{4a(1 - \cos^3(t))}{3\sin^2(t)}.$$

Show further that

$$L(t) = \frac{4a}{3}\left(\cos(t) + \frac{1}{2}\sec^2\left(\frac{1}{2}\right)\right).$$

(iii) The random variable T is uniformly distributed on $[0, \frac{1}{2}\pi]$. Find the expected value of $L(T)$.