There are 12 questions in this paper.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12

Sixth Term Examination Paper

19-S3

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Section A: Pure Mathematics

1 The coordinates of a particle at time t are x and y. For $t \geq 0$, they satisfy the pair of coupled differential equations

$$\dot{x} = -x - ky$$
$$\dot{y} = x - y$$

where k is a constant. When $t = 0, x = 1$ and $y = 0$.

(i) Let $k = 1$. Find x and y in terms of t and sketch y as a function of t.

Sketch the path of the particle in the x - y plane, giving the coordinates of the point at which y is greatest and the coordinates of the point at which x is least.

(ii) Instead, let $k = 0$. Find x and y in terms of t and sketch the path of the particle in the x - y plane.

2 The definition of the derivative f' of a (differentiable) function f is

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}. \qquad (*)$$

(i) The function f has derivative f' and satisfies

$$\mathrm{f}(x+y) = \mathrm{f}(x)\mathrm{f}(y)$$

for all x and y, and $\mathrm{f}'(0) = k$ where $k \neq 0$. Show that $\mathrm{f}(0) = 1$.

Using (*), show that $\mathrm{f}'(x) = k\mathrm{f}(x)$ and find $\mathrm{f}(x)$ in terms of x and k.

(ii) The function g has derivative g' and satisfies

$$\mathrm{g}(x+y) = \frac{\mathrm{g}(x) + \mathrm{g}(y)}{1 + \mathrm{g}(x)\mathrm{g}(y)}$$

for all x and y, $|\mathrm{g}(x)| < 1$ for all x, and $\mathrm{g}'(0) = k$ where $k \neq 0$.

Find $\mathrm{g}'(x)$ in terms of $\mathrm{g}(x)$ and k, and hence find $\mathrm{g}(x)$ in terms of x and k.

3 The matrix A is given by

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(i) You are given that the transformation represented by $\mathbf{A}$ has a line L_1 of invariant points (so that each point on L_1 is transformed to itself). Let (x, y) be a point on L_1. Show that $((a-1)(d-1)-bc)xy = 0$.

Show further that $(a-1)(d-1) = bc$.

What can be said about $\mathbf{A}$ if L_1 does not pass through the origin?

(ii) By considering the cases $b \neq 0$ and $b = 0$ separately, show that if $(a-1)(d-1) = bc$ then the transformation represented by $\mathbf{A}$ has a line of invariant points. You should identify the line in the different cases that arise.

(iii) You are given instead that the transformation represented by $\mathbf{A}$ has an invariant line L_2 (so that each point on L_2 is transformed to a point on L_2) and that L_2 does not pass through the origin. If L_2 has the form $y = mx + k$, show that $(a-1)(d-1) = bc$.

4 The n^{th} degree polynomial $\mathrm{P}(x)$ is said to be *reflexive* if:

(a) $\mathrm{P}(x)$ is of the form $x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n$ where $n \geq 1$;

(b) $a_1, a_2, \ldots, a_n$ are real;

(c) the n (not necessarily distinct) roots of the equation $\mathrm{P}(x) = 0$ are $a_1, a_2, \ldots, a_n$.

(i) Find all reflexive polynomials of degree less than or equal to 3.

(ii) For a reflexive polynomial with $n > 3$, show that

$$2a_2 = -a_2^2 - a_3^2 - \cdots - a_n^2.$$

Deduce that, if all the coefficients of a reflexive polynomial of degree n are integers and $a_n \neq 0$, then $n \leq 3$.

(iii) Determine all reflexive polynomials with integer coefficients.

5 **(i)** Let

$$f(x) = \frac{x}{\sqrt{x^2 + p}},$$

where p is a non- zero constant. Sketch the curve $y = f(x)$ for $x \geq 0$ in the case $p > 0$.

(ii) Let

$$I = \int \frac{1}{(b^2 - y^2)\sqrt{c^2 - y^2}}\,\mathrm{d}y,$$

where b and c are positive constants. Use the substitution $y = \dfrac{cx}{\sqrt{x^2 + p}}$, where p is a suitably chosen constant, to show that

$$I = \int \frac{1}{b^2 + (b^2 - c^2)x^2}\,\mathrm{d}x.$$

Evaluate

$$\int_1^{\sqrt{2}} \frac{1}{(3 - y^2)\sqrt{2 - y^2}}\,\mathrm{d}y.$$

Note:

$$\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + \text{constant}.$$

Hence evaluate

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{y}{(3y^2 - 1)\sqrt{2y^2 - 1}}\,\mathrm{d}y.$$

(iii) By means of a suitable substitution, evaluate

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{(3y^2 - 1)\sqrt{2y^2 - 1}}\,\mathrm{d}y.$$

6 The point P in the Argand diagram is represented by the the complex number z, which satisfies

$$zz^* - az^* - a^*z + aa^* - r^2 = 0.$$

Here, r is a positive real number and $r^2 \neq a^*a$. By writing $|z-a|^2$ as $(z-a)(z-a)^*$, show that the locus of P is a circle, C, the radius and the centre of which you should give.

(i) The point Q is represented by ω, and is related to P by $\omega = \frac{1}{z}$. Let C' be the locus of Q. Show that C' is also a circle, and give its radius and centre.

If C and C' are the same circle, show that

$$\left(|a|^2 - r^2\right)^2 = 1$$

and that either a is real or a is imaginary. Give sketches to indicate the position of C in these two cases.

(ii) Suppose instead that the point Q is represented by ω, where $\omega = \frac{1}{z*}$. If the locus of Q is C, is it the case that either a is real or a is imaginary?

7 The *Devil's Curve* is given by

$$y^2(y^2 - b^2) = x^2(x^2 - a^2),$$

where a and b are positive constants.

(i) In the case $a = b$, sketch the Devil's Curve.

(ii) Now consider the case $a = 2$ and $b = \sqrt{5}$, and $x \geq 0, y \geq 0$.

(a) Show by considering a quadratic equation in x^2 that either $0 \leqslant y \leqslant 1$ or $y \geqslant 2$.

(b) Describe the curve very close to and very far from the origin.

(c) Find the points at which the tangent to the curve is parallel to the x-axis and the point at which the tangent to the curve is parallel to the y-axis.

Sketch the Devil's Curve in this case.

(iii) Sketch the Devil's Curve in the case $a = 2$ and $b = \sqrt{5}$ again, but with $-\infty < x < \infty$ and $-\infty < y < \infty$.

8 A pyramid has a horizontal rectangular base $ABCD$ and its vertex V is vertically above the centre of the base. The acute angle between the face AVB and the base is α, the acute angle between the face BVC and the base is β and the obtuse angle between the faces AVB and BVC is $\pi - \theta$.

(i) The edges AB and BC are parallel to the unit vectors $\mathbf{i}$ and $\mathbf{j}$, respectively, and the unit vector k is vertical. Find a unit vector that is perpendicular to the face AVB. Show that

$$\cos\theta = \cos\alpha\cos\beta.$$

(ii) The edge BV makes an angle ϕ with the base. Show that

$$\cot^2\phi = \cot^2\alpha + \cot^2\beta.$$

Show also that

$$\cos^2\phi = \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1-\cos^2\theta} \geqslant \frac{2\cos\theta - 2\cos^2\theta}{1-\cos^2\theta}$$

and deduce that $\phi < \theta$.

Section B: Mechanics

9 In this question, $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors and $\mathbf{j}$ is vertically upwards.

A smooth hemisphere of mass M and radius a rests on a smooth horizontal table with its plane face in contact with the table. The point A is at the top of the hemisphere and the point O is at the centre of its plane face.

Initially, a particle P of mass m rests at A. It is then given a small displacement in the positive $\mathbf{i}$ direction. At a later time t, when the particle is still in contact with the hemisphere, the hemisphere has been displaced by $-s\mathbf{i}$ and $\angle AOP = \theta$.

(i) Let r be the position vector of the particle at time t with respect to the initial position of O. Write down an expression for $\mathbf{r}$ in terms of a, θ and s and show that

$$\dot{\mathbf{r}} = \left(a\dot{\theta}\cos\theta - \dot{s}\right)\mathbf{i} - a\dot{\theta}\sin\theta\mathbf{j}.$$

Show also that

$$\dot{s} = (1-k)a\dot{\theta}\cos\theta,$$

where $k = \frac{M}{m+M}$, and deduce that

$$\dot{\mathbf{r}} = a\dot{\theta}\left(k\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\right).$$

(ii) Show that

$$a\dot{\theta}^2\left(k\cos^2\theta + \sin^2\theta\right) = 2g(1-\cos\theta).$$

(iii) At time T, when $\theta = \alpha$, the particle leaves the hemisphere. By considering the component of $\ddot{\mathbf{r}}$ parallel to the vector $\sin\theta\mathbf{i} + k\cos\theta\mathbf{j}$, or otherwise, show that at time T

$$a\dot{\theta}^2 = g\cos\alpha.$$

Find a cubic equation for $\cos\alpha$ and deduce that $\cos\alpha > \frac{2}{3}$.

10 Two identical smooth spheres P and Q can move on a smooth horizontal table. Initially, P moves with speed u and Q is at rest. Then P collides with Q. The direction of travel of P before the collision makes an acute angle α with the line joining the centres of P and Q at the moment of the collision. The coefficient of restitution between P and Q is e where $e < 1$. As a result of the collision, P has speed v and Q has speed w, and P is deflected through an angle θ.

(i) Show that

$$u\sin\alpha = v\sin(\alpha+\theta)$$

and find an expression for w in terms of v, θ and α.

(ii) Show further that

$$\sin\theta = \cos(\theta+\alpha)\sin\alpha + e\sin(\theta+\alpha)\cos\alpha$$

and find an expression for $\tan\theta$ in terms of $\tan\alpha$ and e.

Find, in terms of e, the maximum value of $\tan\theta$ as α varies.

Section C: Probability and Statistics

11 The number of customers arriving at a builders' merchants each day follows a Poisson distribution with mean λ. Each customer is offered some free sand. The probability of any given customer taking the free sand is p.

(i) Show that the number of customers each day who take sand follows a Poisson distribution with mean $p\lambda$.

(ii) The merchant has a mass S of sand at the beginning of the day. Each customer who takes the free sand gets a proportion k of the remaining sand, where $0 \leqslant k < 1$. Show that by the end of the day the expected mass of sand taken is

$$(1 - \mathrm{e}^{-kp\lambda})S.$$

(iii) At the beginning of the day, the merchant's bag of sand contains a large number of grains, exactly one of which is made from solid gold. At the end of the day, the merchant's assistant takes a proportion k of the remaining sand. Find the probability that the assistant takes the golden grain. Comment on the case $k = 0$ and on the limit $k \to 1$.

In the case $p\lambda > 1$, find the value of k which maximises the probability that the assistant takes the golden grain.

12 The set S is the set of all integers from 1 to n. The set T is the set of all distinct subsets of S, including the empty set $\varnothing$ and S itself. Show that T contains exactly 2^n sets.

The sets $A_1, A_2, \ldots, A_m$, which are not necessarily distinct, are chosen randomly and independently from T, and for each $k (1 \leqslant k \leqslant m)$, the set A_k is equally likely to be any of the sets in T.

(i) Write down the value of $\Pr(1 \in A_1)$.

(ii) By considering each integer separately, show that

$$\Pr\left(A_1 \cap A_2 = \varnothing\right) = \left(\frac{3}{4}\right)^n .$$

Find

$$\Pr\left(A_1 \cap A_2 \cap A_3 = \varnothing\right) \quad \text{and} \quad \Pr\left(A_1 \cap A_2 \cap \cdots \cap A_m = \varnothing\right).$$

(iii) Find

$$\Pr(A_1 \subseteq A_2), \Pr(A_1 \subseteq A_2 \subseteq A3) \quad \text{and} \quad \Pr(A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m).$$