

THERE ARE 13 QUESTIONS IN THIS PAPER.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13

Sixth Term Examination Paper

16-S3



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Section A: Pure Mathematics

1 Let

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2 + 2ax + b)^n} dx,$$

where a and b are constants with $b > a^2$, and n is a positive integer.

(i) By using the substitution $x + a = \sqrt{b - a^2} \tan u$, or otherwise, show that

$$I_1 = \frac{\pi}{\sqrt{b - a^2}}.$$

(ii) Show that $2n(b - a^2) I_{n+1} = (2n - 1) I_n$.

(iii) Hence prove by induction that

$$I_n = \frac{\pi}{2^{2n-2}(b - a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.$$

2 The distinct points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ and $R(ar^2, 2ar)$ lie on the parabola $y^2 = 4ax$, where $a > 0$. The points are such that the normal to the parabola at Q and the normal to the parabola at R both pass through P .

(i) Show that $q^2 + qp + 2 = 0$.

(ii) Show that QR passes through a certain point that is independent of the choice of P .

(iii) Let T be the point of intersection of OP and QR , where O is the coordinate origin. Show that T lies on a line that is independent of the choice of P .

Show further that the distance from the x -axis to T is less than $\frac{a}{\sqrt{2}}$.

3 (i) Given that

$$\int \frac{x^3 - 2}{(x + 1)^2} e^x dx = \frac{P(x)}{Q(x)} e^x + \text{constant},$$

where $P(x)$ and $Q(x)$ are polynomials, show that $Q(x)$ has a factor of $x + 1$.

Show also that the degree of $P(x)$ is exactly one more than the degree of $Q(x)$, and find $P(x)$ in the case $Q(x) = x + 1$.

(ii) Show that there are no polynomials $P(x)$ and $Q(x)$ such that

$$\int \frac{1}{x + 1} e^x dx = \frac{P(x)}{Q(x)} e^x + \text{constant}.$$

You need consider only the case when $P(x)$ and $Q(x)$ have no common factors.

4 (i) By considering $\frac{1}{1 + x^r} - \frac{1}{1 + x^{r+1}}$ for $|x| \neq 1$, simplify

$$\sum_{r=1}^N \frac{x^r}{(1 + x^r)(1 + x^{r+1})}.$$

Show that, for $|x| < 1$,

$$\sum_{r=1}^{\infty} \frac{x^r}{(1 + x^r)(1 + x^{r+1})} = \frac{x}{1 - x^2}.$$

(ii) Deduce that

$$\sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r + 1)y) = 2e^{-y} \operatorname{cosech}(2y)$$

for $y > 0$.

Hence simplify

$$\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r + 1)y),$$

for $y > 0$.

- 5 (i) By considering the binomial expansion of $(1+x)^{2m+1}$, prove that

$$\binom{2m+1}{m} < 2^{2m},$$

for any positive integer m .

- (ii) For any positive integers r and s with $r < s$, $P_{r,s}$ is defined as follows: $P_{r,s}$ is the product of all the prime numbers greater than r and less than or equal to s , if there are any such primes numbers; if there are no such primes numbers, then $P_{r,s} = 1$.

For example, $P_{3,7} = 35$, $P_{7,10} = 1$ and $P_{14,18} = 17$.

Show that, for any positive integer m , $P_{m+1,2m+1}$ divides $\binom{2m+1}{m}$, and deduce that

$$P_{m+1,2m+1} < 2^{2m}.$$

- (iii) Show that, if $P_{1,k} < 4^k$ for $k = 2, 3, \dots, 2m$, then $P_{1,2m+1} < 4^{2m+1}$.

- (iv) Prove that $P_{1,n} < 4^n$ for $n \geq 2$.

- 6 Show, by finding R and γ , that $A \sinh x + B \cosh x$ can be written in the form $R \cosh(x + \gamma)$ if $B > A > 0$. Determine the corresponding forms in the other cases that arise, for $A > 0$, according to the value of B .

Two curves have equations $y = \operatorname{sech} x$ and $y = a \tanh x + b$, where $a > 0$.

- (i) In the case $b > a$, show that if the curves intersect then the x -coordinates of the points of intersection can be written in the form

$$\pm \operatorname{arcosh} \left(\frac{1}{\sqrt{b^2 - a^2}} \right) - \operatorname{artanh} \frac{a}{b}.$$

- (ii) Find the corresponding result in the case $a > b > 0$.

- (iii) Find necessary and sufficient conditions on a and b for the curves to intersect at two distinct points.

- (iv) Find necessary and sufficient conditions on a and b for the curves to touch and, given that they touch, express the y -coordinate of the point of contact in terms of a .

- 7 Let $\omega = e^{2\pi i/n}$, where n is a positive integer. Show that, for any complex number z ,

$$(z - 1)(z - \omega) \cdots (z - \omega^{n-1}) = z^n - 1.$$

The points X_0, X_1, \dots, X_{n-1} lie on a circle with centre O and radius 1, and are the vertices of a regular polygon.

- (i) The point P is equidistant from X_0 and X_1 . Show that, if n is even,

$$|PX_0| \times |PX_1| \times \cdots \times |PX_{n-1}| = |OP|^n + 1,$$

where $|PX_k|$ denotes the distance from P to X_k .

Give the corresponding result when n is odd. (There are two cases to consider.)

- (ii) Show that

$$|X_0X_1| \times |X_0X_2| \times \cdots \times |X_0X_{n-1}| = n.$$

- 8 (i) The function f satisfies, for all x , the equation

$$f(x) + (1 - x)f(-x) = x^2.$$

Show that $f(-x) + (1 + x)f(x) = x^2$. Hence find $f(x)$ in terms of x . You should verify that your function satisfies the original equation.

- (ii) The function K is defined, for $x \neq 1$, by

$$K(x) = \frac{x + 1}{x - 1}.$$

Show that, for $x \neq 1$, $K(K(x)) = x$.

The function g satisfies the equation

$$g(x) + xg\left(\frac{x + 1}{x - 1}\right) = x \quad (x \neq 1).$$

Show that, for $x \neq 1$, $g(x) = \frac{2x}{x^2 + 1}$.

- (iii) Find $h(x)$, for $x \neq 0, x \neq 1$, given that

$$h(x) + h\left(\frac{1}{1 - x}\right) = 1 - x - \frac{1}{1 - x} \quad (x \neq 0, x \neq 1).$$

Section B: Mechanics

- 9** Three pegs P, Q and R are fixed on a smooth horizontal table in such a way that they form the vertices of an equilateral triangle of side $2a$. A particle X of mass m lies on the table. It is attached to the pegs by three springs, PX, QX and RX , each of modulus of elasticity λ and natural length l , where $l < \frac{2}{\sqrt{3}}a$. Initially the particle is in equilibrium.

- (i) Show that the extension in each spring is $\frac{2}{\sqrt{3}}a - l$.
- (ii) The particle is then pulled a small distance directly towards P and released. Show that the tension T in the spring RX is given by

$$T = \frac{\lambda}{l} \left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} - l \right),$$

where x is the displacement of X from its equilibrium position.

- (iii) Show further that the particle performs approximate simple harmonic motion with period

$$2\pi \sqrt{\frac{4mla}{3(4a - \sqrt{3}l)\lambda}}.$$

- 10** A smooth plane is inclined at an angle α to the horizontal. A particle P of mass m is attached to a fixed point A above the plane by a light inextensible string of length a . The particle rests in equilibrium on the plane, and the string makes an angle β with the plane.

The particle is given a horizontal impulse parallel to the plane so that it has an initial speed of u .

- (i) Show that the particle will not immediately leave the plane if $ag \cos(\alpha + \beta) > u^2 \tan \beta$.
- (ii) Show further that a necessary condition for the particle to perform a complete circle whilst in contact with the plane is $6 \tan \alpha \tan \beta < 1$.

- 11** A car of mass m travels along a straight horizontal road with its engine working at a constant rate P . The resistance to its motion is such that the acceleration of the car is zero when it is moving with speed $4U$.

- (i) Given that the resistance is proportional to the car's speed, show that the distance X_1 travelled by the car while it accelerates from speed U to speed $2U$, is given by

$$\lambda X_1 = 2 \ln \frac{9}{5} - 1,$$

where $\lambda = P/(16mU^3)$.

- (ii) Given instead that the resistance is proportional to the square of the car's speed, show that the distance X_2 travelled by the car while it accelerates from speed U to speed $2U$ is given by

$$\lambda X_2 = \frac{4}{3} \ln \frac{9}{8}.$$

- (iii) Given that $3.17 < \ln 24 < 3.18$ and $1.60 < \ln 5 < 1.61$, determine which is the larger of X_1 and X_2 .

Section C: Probability and Statistics

- 12** Let X be a random variable with mean μ and standard deviation σ . *Chebyshev's inequality*, which you may use without proof, is

$$P(|X - \mu| > k\sigma) \leq \frac{1}{k^2},$$

where k is any positive number.

- (i) The probability of a biased coin landing heads up is 0.2. It is thrown $100n$ times, where n is an integer greater than 1. Let α be the probability that the coin lands heads up N times, where $16n \leq N \leq 24n$.

Use Chebyshev's inequality to show that

$$\alpha \geq 1 - \frac{1}{n}.$$

- (ii) Use Chebyshev's inequality to show that

$$1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{2n}}{(2n)!} \geq \left(1 - \frac{1}{n}\right) e^n.$$

- 13** Given a random variable X with mean μ and standard deviation σ , we define the *kurtosis*, κ , of X by

$$\kappa = \frac{E((X - \mu)^4)}{\sigma^4} - 3.$$

Show that the random variable $X - a$, where a is a constant, has the same kurtosis as X .

- (i) Show by integration that a random variable which is Normally distributed with mean 0 has kurtosis 0.

- (ii) Let Y_1, Y_2, \dots, Y_n be n independent, identically distributed, random variables with mean 0, and let $T = \sum_{r=1}^n Y_r$. Show that

$$E(T^4) = \sum_{r=1}^n E(Y_r^4) + 6 \sum_{r=1}^{n-1} \sum_{s=r+1}^n E(Y_s^2)E(Y_r^2).$$

- (iii) Let X_1, X_2, \dots, X_n be n independent, identically distributed, random variables each with kurtosis κ . Show that the kurtosis of their sum is $\frac{\kappa}{n}$.