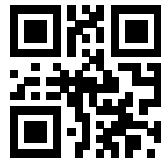


THERE ARE 13 QUESTIONS IN THIS PAPER.

Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13

Sixth Term Examination Paper

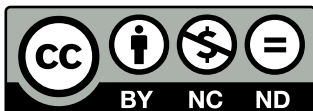
11-S1



Compiled by: Dr Yu 郁博士

www.CasperYC.club

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SUGGESTIONS TO DRYUFROMSHANGHAI@QQ.COM

Section A: Pure Mathematics

- 1 (i) Show that the gradient of the curve $\frac{a}{x} + \frac{b}{y} = 1$, where $b \neq 0$, is $-\frac{ay^2}{bx^2}$.

The point (p, q) lies on both the straight line $ax + by = 1$ and the curve $\frac{a}{x} + \frac{b}{y} = 1$, where $ab \neq 0$.
Given that, at this point, the line and the curve have the same gradient, show that $p = \pm q$.

Show further that either $(a - b)^2 = 1$ or $(a + b)^2 = 1$.

- (ii) Show that if the straight line $ax + by = 1$, where $ab \neq 0$, is a normal to the curve $\frac{a}{x} - \frac{b}{y} = 1$, then $a^2 - b^2 = \frac{1}{2}$.

- 2 The number E is defined by $E = \int_0^1 \frac{e^x}{1+x} dx$.

Show that

$$\int_0^1 \frac{xe^x}{1+x} dx = e - 1 - E,$$

and evaluate $\int_0^1 \frac{x^2 e^x}{1+x} dx$ in terms of e and E .

Evaluate also, in terms of E and e as appropriate:

(i) $\int_0^1 \frac{e^{\frac{1-x}{1+x}}}{1+x} dx;$

(ii) $\int_1^{\sqrt{2}} \frac{e^{x^2}}{x} dx.$

3 Prove the identity

$$4 \sin \theta \sin\left(\frac{1}{3}\pi - \theta\right) \sin\left(\frac{1}{3}\pi + \theta\right) = \sin 3\theta. \quad (*)$$

(i) By differentiating (*), or otherwise, show that

$$\cot \frac{1}{9}\pi - \cot \frac{2}{9}\pi + \cot \frac{4}{9}\pi = \sqrt{3}.$$

(ii) By setting $\theta = \frac{1}{6}\pi - \phi$ in (*), or otherwise, obtain a similar identity for $\cos 3\theta$ and deduce that

$$\cot \theta \cot\left(\frac{1}{3}\pi - \theta\right) \cot\left(\frac{1}{3}\pi + \theta\right) = \cot 3\theta.$$

Show that

$$\operatorname{cosec} \frac{1}{9}\pi - \operatorname{cosec} \frac{5}{9}\pi + \operatorname{cosec} \frac{7}{9}\pi = 2\sqrt{3}.$$

4 The distinct points P and Q , with coordinates $(ap^2, 2ap)$ and $(aq^2, 2aq)$ respectively, lie on the curve $y^2 = 4ax$. The tangents to the curve at P and Q meet at the point T .

(i) Show that T has coordinates $(apq, a(p+q))$. You may assume that $p \neq 0$ and $q \neq 0$.

(ii) The point F has coordinates $(a, 0)$ and ϕ is the angle TFP . Show that

$$\cos \phi = \frac{pq + 1}{\sqrt{(p^2 + 1)(q^2 + 1)}}$$

and deduce that the line FT bisects the angle PFQ .

- 5 (i) Given that $0 < k < 1$, show with the help of a sketch that the equation

$$\sin x = kx \quad (*)$$

has a unique solution in the range $0 < x < \pi$.

- (ii) Let

$$I = \int_0^\pi |\sin x - kx| \, dx.$$

Show that

$$I = \frac{\pi^2 \sin \alpha}{2\alpha} - 2 \cos \alpha - \alpha \sin \alpha,$$

where α is the unique solution of $(*)$.

- (iii) Show that I , regarded as a function of α , has a unique stationary value and that this stationary value is a minimum.

- (iv) Deduce that the smallest value of I is

$$-2 \cos \frac{\pi}{\sqrt{2}}.$$

- 6 Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

- (i) Show that the coefficient of x^r in the expansion of

$$\frac{1-x+2x^2}{(1-x)^3}$$

is $r^2 + 1$ and hence find the sum of the series

$$1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \frac{26}{32} + \frac{37}{64} + \frac{50}{128} + \dots$$

- (ii) Find the sum of the series

$$1 + 2 + \frac{9}{4} + 2 + \frac{25}{16} + \frac{9}{8} + \frac{49}{64} + \dots$$

- 7** In this question, you may assume that $\ln(1+x) \approx x - \frac{1}{2}x^2$ when $|x|$ is small.

The height of the water in a tank at time t is h . The initial height of the water is H and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.

- (i)** Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that

$$\frac{dh}{dt} = k(\alpha^2 H - h),$$

for some positive constant k . Deduce that the time T taken for the water to reach height αH is given by

$$kT = \ln \left(1 + \frac{1}{\alpha} \right),$$

and that $kT \approx \alpha^{-1}$ for large values of α .

- (ii)** Suppose that the rate at which water leaks out of the tank is proportional to \sqrt{h} (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by

$$cT' = 2\sqrt{H} \left(1 - \sqrt{\alpha} + \alpha \ln \left(1 + \frac{1}{\sqrt{\alpha}} \right) \right)$$

for some positive constant c , and that $cT' \approx \sqrt{H}$ for large values of α .

- 8** **(i)** The numbers m and n satisfy

$$m^3 = n^3 + n^2 + 1. \quad (*)$$

- (a)** Show that $m > n$. Show also that $m < n + 1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n + 1$ unless $-\frac{3}{2} \leq n \leq 0$.

- (b)** Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m, n) = (1, 0)$ and $(m, n) = (1, -1)$.

- (ii)** Find all integer solutions of the equation

$$p^3 = q^3 + 2q^2 - 1.$$

Section B: Mechanics

- 9** A particle is projected at an angle θ above the horizontal from a point on a horizontal plane. The particle just passes over two walls that are at horizontal distances d_1 and d_2 from the point of projection and are of heights d_2 and d_1 , respectively. Show that

$$\tan \theta = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2}.$$

Find (and simplify) an expression in terms of d_1 and d_2 only for the range of the particle.

- 10** A particle, A , is dropped from a point P which is at a height h above a horizontal plane. A second particle, B , is dropped from P and first collides with A after A has bounced on the plane and before A reaches P again. The bounce and the collision are both perfectly elastic.

- (i) Explain why the speeds of A and B immediately before the first collision are the same.
- (ii) The masses of A and B are M and m , respectively, where $M > 3m$, and the speed of the particles immediately before the first collision is u . Show that both particles move upwards after their first collision and that the maximum height of B above the plane after the first collision and before the second collision is

$$h + \frac{4M(M - m)u^2}{(M + m)^2 g}.$$

- 11** A thin non-uniform bar AB of length $7d$ has centre of mass at a point G , where $AG = 3d$. A light inextensible string has one end attached to A and the other end attached to B . The string is hung over a smooth peg P and the bar hangs freely in equilibrium with B lower than A .

- (i) Show that

$$3 \sin \alpha = 4 \sin \beta,$$

where α and β are the angles PAB and PBA , respectively.

- (ii) Given that $\cos \beta = \frac{4}{5}$ and that α is acute, find in terms of d the length of the string and show that the angle of inclination of the bar to the horizontal is $\arctan \frac{1}{7}$.

Section C: Probability and Statistics

12 I am selling raffle tickets for £1 per ticket. In the queue for tickets, there are m people each with a single £1 coin and n people each with a single £2 coin. Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur. Initially, I have no coins and a large supply of tickets. I stop selling tickets if I cannot give the required change.

- (i) In the case $n = 1$ and $m \geq 1$, find the probability that I am able to sell one ticket to each person in the queue.
- (ii) By considering the first three people in the queue, show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 2$ and $m \geq 2$ is $\frac{m-1}{m+1}$.
- (iii) Show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 3$ and $m \geq 3$ is $\frac{m-2}{m+1}$.

13 In this question, you may use without proof the following result:

$$\int \sqrt{4-x^2} \, dx = 2 \arcsin\left(\frac{1}{2}x\right) + \frac{1}{2}x\sqrt{4-x^2} + c.$$

A random variable X has probability density function f given by

$$f(x) = \begin{cases} 2k & -a \leq x < 0 \\ k\sqrt{4-x^2} & 0 \leq x \leq 2 \\ 0 & \text{otherwise,} \end{cases}$$

where k and a are positive constants.

- (i) Find, in terms of a , the mean of X .
- (ii) Let d be the value of X such that $P(X > d) = \frac{1}{10}$.
Show that $d < 0$ if $2a > 9\pi$ and find an expression for d in terms of a in this case.
- (iii) Given that $d = \sqrt{2}$, find a .