1 Integrate with respect to x

a e^x **b** $4e^x$ **c** $\frac{1}{x}$ **d** $\frac{6}{x}$

2 Integrate with respect to t

a $2 + 3e^t$ **b** $t + t^{-1}$ **c** $t^2 - e^t$ **d** $9 - 2t^{-1}$

e $\frac{7}{t} + \sqrt{t}$ **f** $\frac{1}{4}e^t - \frac{1}{t}$ **g** $\frac{1}{3t} + \frac{1}{t^2}$ **h** $\frac{2}{5t} - \frac{3e^t}{7}$

3 Find

a $\int (5 - \frac{3}{x})\ dx$ **b** $\int (u^{-1} + u^{-2})\ du$ **c** $\int \frac{2e^t + 1}{5}\ dt$

d $\int \frac{3y+1}{y}\ dy$ **e** $\int (\frac{3}{4}e^t + 3\sqrt{t}\ )\ dt$ **f** $\int (x - \frac{1}{x})^2\ dx$

4 The curve $y = f(x)$ passes through the point $(1, -3)$.

Given that $f'(x) = \frac{(2x-1)^2}{x}$, find an expression for $f(x)$.

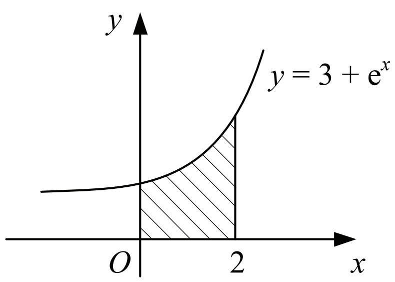
5 Evaluate

a $\int_0^1 (e^x + 10)\ dx$ **b** $\int_2^5 (t + \frac{1}{t})\ dt$ **c** $\int_1^4 \frac{5 - x^2}{x}\ dx$

d $\int_{-2}^{-1} \frac{6y+1}{3y}\ dy$ **e** $\int_{-3}^{3} (e^x - x^2)\ dx$ **f** $\int_2^3 \frac{4r^2 - 3r + 6}{r^2}\ dr$

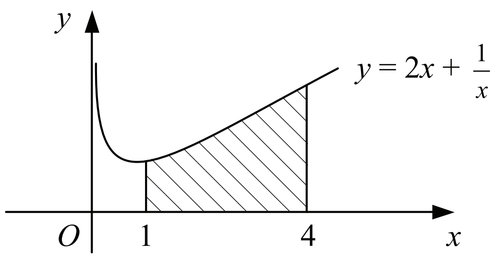
g $\int_{\ln 2}^{\ln 4} (7 - e^u)\ du$ **h** $\int_6^{10} r^{-\frac{1}{2}}(2r^{\frac{1}{2}} + 9r^{-\frac{1}{2}})\ dr$ **i** $\int_4^9 (\frac{1}{\sqrt{x}} + 3e^x)\ dx$

6



The shaded region on the diagram is bounded by the curve $y = 3 + e^x$, the coordinate axes and the line $x = 2$. Show that the area of the shaded region is $e^2 + 5$.

7



The shaded region on the diagram is bounded by the curve $y = 2x + \frac{1}{x}$, the x-axis and the lines $x = 1$ and $x = 4$. Find the area of the shaded region in the form $a + b \ln 2$.

8 Find the exact area of the region enclosed by the given curve, the x-axis and the given ordinates. In each case, $y > 0$ over the interval being considered.

a $y = 4x + 2e^x$, $x = 0$, $x = 1$

b $y = 1 + \frac{3}{x}$, $x = 2$, $x = 4$

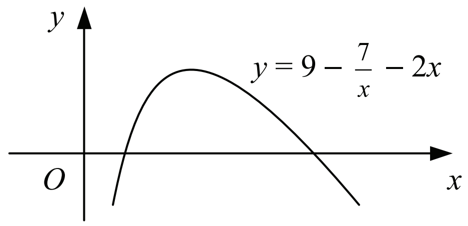
c $y = 4 - \frac{1}{x}$, $x = -3$, $x = -1$

d $y = 2 - \frac{1}{2}e^x$, $x = 0$, $x = \ln 2$

e $y = e^x + \frac{5}{x}$, $x = \frac{1}{2}$, $x = 2$

f $y = \frac{x^3 - 2}{x}$, $x = 2$, $x = 3$

9



The diagram shows the curve with equation $y = 9 - \frac{7}{x} - 2x$, $x > 0$.

a Find the coordinates of the points where the curve crosses the x-axis.

b Show that the area of the region bounded by the curve and the x-axis is $11\frac{1}{4} - 7\ln\frac{7}{2}$.

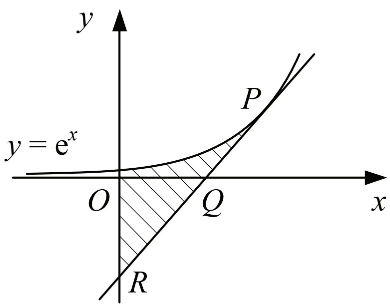
10 **a** Sketch the curve $y = e^x - a$ where a is a constant and $a > 1$.

Show on your sketch the coordinates of any points of intersection with the coordinate axes and the equation of any asymptotes.

b Find, in terms of a, the area of the finite region bounded by the curve $y = e^x - a$ and the coordinate axes.

c Given that the area of this region is $1 + a$, show that $a = e^2$.

11



The diagram shows the curve with equation $y = e^x$. The point P on the curve has x-coordinate 3, and the tangent to the curve at P intersects the x-axis at Q and the y-axis at R.

a Find an equation of the tangent to the curve at P.

b Find the coordinates of the points Q and R.

The shaded region is bounded by the curve, the tangent to the curve at P and the y-axis.

c Find the exact area of the shaded region.

12

$$f(x) \equiv \left(\frac{3}{\sqrt{x}} - 4\right)^2, \ x \in \mathbb{R}, \ x > 0.$$

a Find the coordinates of the point where the curve $y = f(x)$ meets the x-axis.

The finite region R is bounded by the curve $y = f(x)$, the line $x = 1$ and the x-axis.

b Show that the area of R is approximately 0.178

C4 INTEGRATION Worksheet B

1 Integrate with respect to x

a $(x-2)^7$ **b** $(2x+5)^3$ **c** $6(1+3x)^4$ **d** $(\frac{1}{4}x-2)^5$

e $(8-5x)^4$ **f** $\dfrac{1}{(x+7)^2}$ **g** $\dfrac{8}{(4x-3)^5}$ **h** $\dfrac{1}{2(5-3x)^3}$

2 Find

a $\int (3+t)^{\frac{3}{2}}\ \mathrm{d}t$ **b** $\int \sqrt{4x-1}\ \mathrm{d}x$ **c** $\int \dfrac{1}{2y+1}\ \mathrm{d}y$

d $\int \mathrm{e}^{2x-3}\ \mathrm{d}x$ **e** $\int \dfrac{3}{2-7r}\ \mathrm{d}r$ **f** $\int \sqrt[3]{5t-2}\ \mathrm{d}t$

g $\int \dfrac{1}{\sqrt{6-y}}\ \mathrm{d}y$ **h** $\int 5\mathrm{e}^{7-3t}\ \mathrm{d}t$ **i** $\int \dfrac{4}{3u+1}\ \mathrm{d}u$

3 Given $\mathrm{f}'(x)$ and a point on the curve $y = \mathrm{f}(x)$, find an expression for $\mathrm{f}(x)$ in each case.

a $\mathrm{f}'(x) = 8(2x-3)^3$, $(2, 6)$ **b** $\mathrm{f}'(x) = 6\mathrm{e}^{2x+4}$, $(-2, 1)$

c $\mathrm{f}'(x) = 2 - \dfrac{8}{4x-1}$, $(\frac{1}{2}, 4)$ **d** $\mathrm{f}'(x) = 8x - \dfrac{3}{(3x-2)^2}$, $(-1, 3)$

4 Evaluate

a $\int_0^1 (3x+1)^2\ \mathrm{d}x$ **b** $\int_1^2 (2x-1)^3\ \mathrm{d}x$ **c** $\int_2^4 \dfrac{1}{(5-x)^2}\ \mathrm{d}x$

d $\int_{-1}^1 \mathrm{e}^{2x+2}\ \mathrm{d}x$ **e** $\int_2^6 \sqrt{3x-2}\ \mathrm{d}x$ **f** $\int_1^2 \dfrac{4}{6x-3}\ \mathrm{d}x$

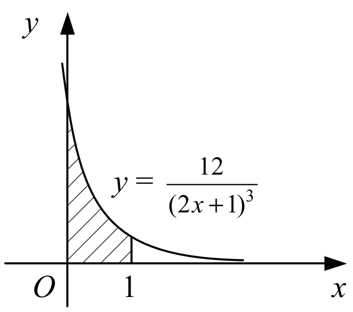
g $\int_0^1 \dfrac{1}{\sqrt[3]{7x+1}}\ \mathrm{d}x$ **h** $\int_{-7}^{-1} \dfrac{1}{5x+3}\ \mathrm{d}x$ **i** $\int_4^7 \left(\dfrac{x-4}{2}\right)^3\ \mathrm{d}x$

5 Find the exact area of the region enclosed by the given curve, the x-axis and the given ordinates. In each case, $y > 0$ over the interval being considered.

a $y = \mathrm{e}^{3-x}$, $x = 3$, $x = 4$ **b** $y = (3x-5)^3$, $x = 2$, $x = 3$

c $y = \dfrac{3}{4x+2}$, $x = 1$, $x = 4$ **d** $y = \dfrac{1}{(1-2x)^2}$, $x = -2$, $x = 0$

6



The diagram shows part of the curve with equation $y = \dfrac{12}{(2x+1)^3}$.

Find the area of the shaded region bounded by the curve, the coordinate axes and the line $x = 1$.

1 **a** Express $\dfrac{3x+5}{(x+1)(x+3)}$ in partial fractions.

b Hence, find $\displaystyle\int \frac{3x+5}{(x+1)(x+3)}\,dx$.

2 Show that $\displaystyle\int \frac{3}{(t-2)(t+1)}\,dt = \ln\left|\frac{t-2}{t+1}\right| + c$.

3 Integrate with respect to x

a $\dfrac{6x-11}{(2x+1)(x-3)}$ **b** $\dfrac{14-x}{x^2+2x-8}$ **c** $\dfrac{6}{(2+x)(1-x)}$ **d** $\dfrac{x+1}{5x^2-14x+8}$

4 **a** Find the values of the constants A, B and C such that

$$\frac{x^2-6}{(x+4)(x-1)} \equiv A + \frac{B}{x+4} + \frac{C}{x-1}.$$

b Hence, find $\displaystyle\int \frac{x^2-6}{(x+4)(x-1)}\,dx$.

5 **a** Express $\dfrac{x^2-x-4}{(x+2)(x+1)^2}$ in partial fractions.

b Hence, find $\displaystyle\int \frac{x^2-x-4}{(x+2)(x+1)^2}\,dx$.

6 Integrate with respect to x

a $\dfrac{3x^2-5}{x^2-1}$ **b** $\dfrac{x(4x+13)}{(2+x)^2(3-x)}$ **c** $\dfrac{x^2-x+1}{x^2-3x-10}$ **d** $\dfrac{2-6x+5x^2}{x^2(1-2x)}$

7 Show that $\displaystyle\int_3^4 \frac{3x-5}{(x-1)(x-2)}\,dx = 2\ln 3 - \ln 2$.

8 Find the exact value of

a $\displaystyle\int_1^3 \frac{x+3}{x(x+1)}\,dx$ **b** $\displaystyle\int_0^2 \frac{3x-2}{x^2+x-12}\,dx$ **c** $\displaystyle\int_1^2 \frac{9}{2x^2-7x-4}\,dx$

d $\displaystyle\int_0^2 \frac{2x^2-7x+7}{x^2-2x-3}\,dx$ **e** $\displaystyle\int_0^1 \frac{5x+7}{(x+1)^2(x+3)}\,dx$ **f** $\displaystyle\int_{-1}^1 \frac{2+x}{8-2x-x^2}\,dx$

9 **a** Express $\dfrac{1}{x^2-a^2}$, where a is a positive constant, in partial fractions.

b Hence, show that $\displaystyle\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$.

c Find $\displaystyle\int \frac{1}{a^2-x^2}\,dx$.

10 Evaluate

a $\displaystyle\int_{-1}^1 \frac{1}{x^2-9}\,dx$ **b** $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{4}{1-x^2}\,dx$ **c** $\displaystyle\int_0^1 \frac{3}{2x^2-8}\,dx$

1 Integrate with respect to x

a $2\cos x$ **b** $\sin 4x$ **c** $\cos \frac{1}{2}x$ **d** $\sin (x + \frac{\pi}{4})$

e $\cos (2x - 1)$ **f** $3\sin (\frac{\pi}{3} - x)$ **g** $\sec x \tan x$ **h** $\operatorname{cosec}^2 x$

i $5\sec^2 2x$ **j** $\operatorname{cosec} \frac{1}{4}x \cot \frac{1}{4}x$ **k** $\frac{4}{\sin^2 x}$ **l** $\frac{1}{\cos^2(4x+1)}$

2 Evaluate

a $\int_0^{\frac{\pi}{2}} \cos x \, dx$ **b** $\int_0^{\frac{\pi}{6}} \sin 2x \, dx$ **c** $\int_0^{\frac{\pi}{2}} 2\sec \frac{1}{2}x \tan \frac{1}{2}x \, dx$

d $\int_0^{\frac{\pi}{3}} \cos (2x - \frac{\pi}{3}) \, dx$ **e** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 3x \, dx$ **f** $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \operatorname{cosec} x \cot x \, dx$

3 **a** Express $\tan^2 \theta$ in terms of $\sec \theta$.

b Show that $\int \tan^2 x \, dx = \tan x - x + c$.

4 **a** Use the identity for $\cos (A + B)$ to express $\cos^2 A$ in terms of $\cos 2A$.

b Find $\int \cos^2 x \, dx$.

5 Find

a $\int \sin^2 x \, dx$ **b** $\int \cot^2 2x \, dx$ **c** $\int \sin x \cos x \, dx$

d $\int \frac{\sin x}{\cos^2 x} \, dx$ **e** $\int 4\cos^2 3x \, dx$ **f** $\int (1 + \sin x)^2 \, dx$

g $\int (\sec x - \tan x)^2 \, dx$ **h** $\int \operatorname{cosec} 2x \cot x \, dx$ **i** $\int \cos^4 x \, dx$

6 Evaluate

a $\int_0^{\frac{\pi}{2}} 2\cos^2 x \, dx$ **b** $\int_0^{\frac{\pi}{4}} \cos 2x \sin 2x \, dx$ **c** $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \tan^2 \frac{1}{2}x \, dx$

d $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos 2x}{\sin^2 2x} \, dx$ **e** $\int_0^{\frac{\pi}{4}} (1 - 2\sin x)^2 \, dx$ **f** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \operatorname{cosec}^2 x \, dx$

7 **a** Use the identities for $\sin (A + B)$ and $\sin (A - B)$ to show that

$$\sin A \cos B \equiv \tfrac{1}{2}[\sin (A + B) + \sin (A - B)].$$

b Find $\int \sin 3x \cos x \, dx$.

8 Integrate with respect to x

a $2\sin 5x \sin x$ **b** $\cos 2x \cos x$ **c** $4\sin x \cos 4x$ **d** $\cos (x + \frac{\pi}{6}) \sin x$

1 Showing your working in full, use the given substitution to find

a $\int 2x(x^2 - 1)^3 \; dx$ $\quad u = x^2 + 1$

b $\int \sin^4 x \cos x \; dx$ $\quad u = \sin x$

c $\int 3x^2(2 + x^3)^2 \; dx$ $\quad u = 2 + x^3$

d $\int 2x e^{x^2} \; dx$ $\quad u = x^2$

e $\int \frac{x}{(x^2+3)^4} \; dx$ $\quad u = x^2 + 3$

f $\int \sin 2x \cos^3 2x \; dx$ $\quad u = \cos 2x$

g $\int \frac{3x}{x^2-2} \; dx$ $\quad u = x^2 - 2$

h $\int x\sqrt{1-x^2} \; dx$ $\quad u = 1 - x^2$

i $\int \sec^3 x \tan x \; dx$ $\quad u = \sec x$

j $\int (x + 1)(x^2 + 2x)^3 \; dx$ $\quad u = x^2 + 2x$

2 **a** Given that $u = x^2 + 3$, find the value of u when

i $x = 0$

ii $x = 1$

b Using the substitution $u = x^2 + 3$, show that

$$\int_0^1 2x(x^2 + 3)^2 \; dx = \int_3^4 u^2 \; du.$$

c Hence, show that

$$\int_0^1 2x(x^2 + 3)^2 \; dx = 12\tfrac{1}{3}.$$

3 Using the given substitution, evaluate

a $\int_1^2 x(x^2 - 3)^3 \; dx$ $\quad u = x^2 - 3$

b $\int_0^{\frac{\pi}{6}} \sin^3 x \cos x \; dx$ $\quad u = \sin x$

c $\int_0^3 \frac{4x}{x^2+1} \; dx$ $\quad u = x^2 + 1$

d $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x \; dx$ $\quad u = \tan x$

e $\int_2^3 \frac{x}{\sqrt{x^2-3}} \; dx$ $\quad u = x^2 - 3$

f $\int_{-2}^{-1} x^2(x^3 + 2)^2 \; dx$ $\quad u = x^3 + 2$

g $\int_0^1 e^{2x}(1 + e^{2x})^3 \; dx$ $\quad u = 1 + e^{2x}$

h $\int_3^5 (x - 2)(x^2 - 4x)^2 \; dx$ $\quad u = x^2 - 4x$

4 **a** Using the substitution $u = 4 - x^2$, show that

$$\int_0^2 x(4 - x^2)^3 \; dx = \int_0^4 \tfrac{1}{2}u^3 \; du.$$

b Hence, evaluate

$$\int_0^2 x(4 - x^2)^3 \; dx.$$

5 Using the given substitution, evaluate

a $\int_0^1 x e^{2-x^2} \; dx$ $\quad u = 2 - x^2$

b $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x} \; dx$ $\quad u = 1 + \cos x$

6 **a** By writing $\cot x$ as $\frac{\cos x}{\sin x}$, use the substitution $u = \sin x$ to show that

$$\int \cot x \; dx = \ln|\sin x| + c.$$

b Show that

$$\int \tan x \; dx = \ln|\sec x| + c.$$

c Evaluate

$$\int_0^{\frac{\pi}{6}} \tan 2x \; dx.$$

7 By recognising a function and its derivative, or by using a suitable substitution, integrate with respect to x

a $3x^2(x^3 - 2)^3$ **b** $e^{\sin x} \cos x$ **c** $\frac{x}{x^2+1}$

d $(2x + 3)(x^2 + 3x)^2$ **e** $x\sqrt{x^2+4}$ **f** $\cot^3 x \operatorname{cosec}^2 x$

g $\frac{e^x}{1+e^x}$ **h** $\frac{\cos 2x}{3+\sin 2x}$ **i** $\frac{x^3}{(x^4-2)^2}$

j $\frac{(\ln x)^3}{x}$ **k** $x^{\frac{1}{2}}(1+x^{\frac{3}{2}})^2$ **l** $\frac{x}{\sqrt{5-x^2}}$

8 Evaluate

a $\int_0^{\frac{\pi}{2}} \sin x\,(1 + \cos x)^2 \; dx$ **b** $\int_{-1}^{0} \frac{e^{2x}}{2-e^{2x}} \; dx$

c $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot x \operatorname{cosec}^4 x \; dx$ **d** $\int_2^4 \frac{x+1}{x^2+2x+8} \; dx$

9 Using the substitution $u = x + 1$, show that

$$\int x(x+1)^3 \; dx = \tfrac{1}{20}(4x-1)(x+1)^4 + c.$$

10 Using the given substitution, find

a $\int x(2x-1)^4 \; dx$ $u = 2x - 1$ **b** $\int x\sqrt{1-x} \; dx$ $u^2 = 1 - x$

c $\int \frac{1}{(1-x^2)^{\frac{3}{2}}} \; dx$ $x = \sin u$ **d** $\int \frac{1}{\sqrt{x}-1} \; dx$ $x = u^2$

e $\int (x+1)(2x+3)^3 \; dx$ $u = 2x + 3$ **f** $\int \frac{x^2}{\sqrt{x-2}} \; dx$ $u^2 = x - 2$

11 Using the given substitution, evaluate

a $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} \; dx$ $x = \sin u$ **b** $\int_0^2 x(2-x)^3 \; dx$ $u = 2 - x$

c $\int_0^1 \sqrt{4-x^2} \; dx$ $x = 2\sin u$ **d** $\int_0^3 \frac{x^2}{x^2+9} \; dx$ $x = 3\tan u$

1 Using integration by parts, show that

$$\int x \cos x \;\, dx = x \sin x + \cos x + c.$$

2 Use integration by parts to find

a $\int xe^x \;\, dx$ **b** $\int 4x \sin x \;\, dx$ **c** $\int x \cos 2x \;\, dx$

d $\int x\sqrt{x+1} \;\, dx$ **e** $\int \frac{x}{e^{3x}} \;\, dx$ **f** $\int x \sec^2 x \;\, dx$

3 Using

i integration by parts,
ii the substitution $u = 2x + 1$,

find $\int x(2x+1)^3 \;\, dx$, and show that your answers are equivalent.

4 Show that

$$\int_0^2 xe^{-x} \;\, dx = 1 - 3e^{-2}.$$

5 Evaluate

a $\int_0^{\frac{\pi}{6}} x \cos x \;\, dx$ **b** $\int_0^1 xe^{2x} \;\, dx$ **c** $\int_0^{\frac{\pi}{4}} x \sin 3x \;\, dx$

6 Using integration by parts twice in each case, show that

a $\int x^2 e^x \;\, dx = e^x(x^2 - 2x + 2) + c,$

b $\int e^x \sin x \;\, dx = \frac{1}{2}e^x(\sin x - \cos x) + c.$

7 Find

a $\int x^2 \sin x \;\, dx$ **b** $\int x^2 e^{3x} \;\, dx$ **c** $\int e^{-x} \cos 2x \;\, dx$

8 **a** Write down the derivative of $\ln x$ with respect to x.

b Use integration by parts to find

$$\int \ln x \;\, dx.$$

9 Find

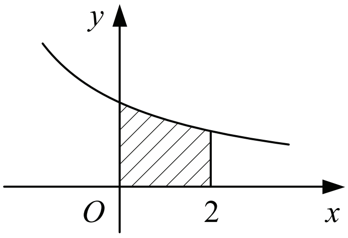
a $\int \ln 2x \;\, dx$ **b** $\int 3x \ln x \;\, dx$ **c** $\int (\ln x)^2 \;\, dx$

10 Evaluate

a $\int_{-1}^0 (x+2)e^x \;\, dx$ **b** $\int_1^2 x^2 \ln x \;\, dx$ **c** $\int_{\frac{1}{3}}^1 2xe^{3x-1} \;\, dx$

d $\int_0^3 \ln(2x+3) \;\, dx$ **e** $\int_0^{\frac{\pi}{2}} x^2 \cos x \;\, dx$ **f** $\int_0^{\frac{\pi}{4}} e^{3x} \sin 2x \;\, dx$

1



The diagram shows part of the curve with parametric equations

$$x = 2t - 4, \quad y = \frac{1}{t}.$$

The shaded region is bounded by the curve, the coordinate axes and the line $x = 2$.

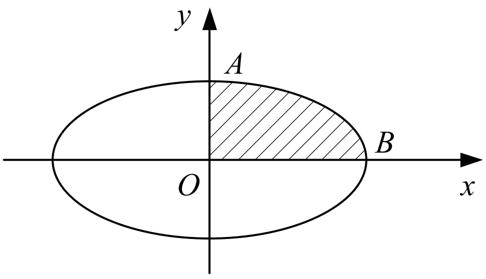
a Find the value of the parameter t when $x = 0$ and when $x = 2$.

b Show that the area of the shaded region is given by $\int_2^3 \frac{2}{t} \, dt$.

c Hence, find the area of the shaded region.

d Verify your answer to part **c** by first finding a cartesian equation for the curve.

2



The diagram shows the ellipse with parametric equations

$$x = 4\cos\theta, \quad y = 2\sin\theta, \quad 0 \le \theta < 2\pi,$$

which meets the positive coordinate axes at the points A and B.

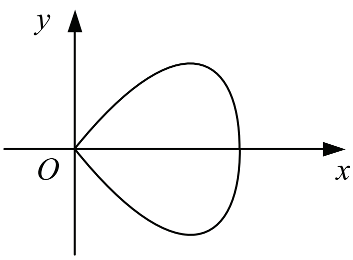
a Find the value of the parameter θ at the points A and B.

b Show that the area of the shaded region bounded by the curve and the positive coordinate axes is given by

$$\int_0^{\frac{\pi}{2}} 8\sin^2\theta \, d\theta.$$

c Hence, show that the area of the region enclosed by the ellipse is 8π.

3



The diagram shows the curve with parametric equations

$$x = 2\sin t, \quad y = 5\sin 2t, \quad 0 \le t < \pi.$$

a Show that the area of the region enclosed by the curve is given by $\int_0^{\frac{\pi}{2}} 20\sin 2t\cos t \, dt$.

b Evaluate this integral.

1 Using an appropriate method, integrate with respect to x

a $(2x - 3)^4$ **b** $\operatorname{cosec}^2 \frac{1}{2}x$ **c** $2e^{4x-1}$ **d** $\frac{2(x-1)}{x(x+1)}$

e $\frac{3}{\cos^2 2x}$ **f** $x(x^2 + 3)^3$ **g** $\sec^4 x \tan x$ **h** $\sqrt{7+2x}$

i xe^{3x} **j** $\frac{x+2}{x^2-2x-3}$ **k** $\frac{1}{4(x+1)^3}$ **l** $\tan^2 3x$

m $4\cos^2 (2x + 1)$ **n** $\frac{3x}{1-x^2}$ **o** $x \sin 2x$ **p** $\frac{x+4}{x+2}$

2 Evaluate

a $\int_1^2 6e^{2x-3}\, dx$ **b** $\int_0^{\frac{\pi}{3}} \tan x\, dx$ **c** $\int_{-2}^2 \frac{2}{x-3}\, dx$

d $\int_2^3 \frac{6+x}{4+3x-x^2}\, dx$ **e** $\int_1^2 (1-2x)^3\, dx$ **f** $\int_0^{\frac{\pi}{3}} \sin^2 x \sin 2x\, dx$

3 Using the given substitution, evaluate

a $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-x^2}}\, dx$ $x = 3\sin u$ **b** $\int_0^1 x(1-3x)^3\, dx$ $u = 1 - 3x$

c $\int_2^{2\sqrt{3}} \frac{1}{4+x^2}\, dx$ $x = 2\tan u$ **d** $\int_{-1}^0 x^2\sqrt{x+1}\, dx$ $u^2 = x + 1$

4 Integrate with respect to x

a $\frac{2}{5-3x}$ **b** $(x+1)\,e^{x^2+2x}$ **c** $\frac{1-x}{2x+1}$ **d** $\sin 3x \cos 2x$

e $3x(x-1)^4$ **f** $\frac{3x^2+6x+2}{x^2+3x+2}$ **g** $\frac{5}{\sqrt[3]{2x-1}}$ **h** $\frac{\cos x}{2+3\sin x}$

i $\frac{x^2}{\sqrt{x^3-1}}$ **j** $(2 - \cot x)^2$ **k** $\frac{6x-5}{(x-1)(2x-1)^2}$ **l** x^2e^{-x}

5 Evaluate

a $\int_2^4 \frac{1}{3x-4}\, dx$ **b** $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^2 x \cot^2 x\, dx$ **c** $\int_0^1 \frac{7-x^2}{(2-x)^2(3-x)}\, dx$

d $\int_0^{\frac{\pi}{2}} x\cos \frac{1}{2}x\, dx$ **e** $\int_1^5 \frac{1}{\sqrt{4x+5}}\, dx$ **f** $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\cos x \cos 3x\, dx$

g $\int_0^2 x\sqrt{2x^2+1}\, dx$ **h** $\int_0^1 \frac{x^2+1}{x-2}\, dx$ **i** $\int_0^1 (x-2)(x+1)^3\, dx$

6 Find the exact area of the region enclosed by the given curve, the x-axis and the given ordinates.

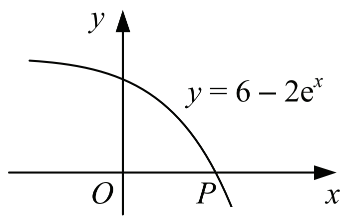
a $y = \frac{x}{(x^2+2)^3}$, $x = 1$, $x = 2$ **b** $y = \ln x$, $x = 2$, $x = 4$

7 Given that

$$\int_3^6 \frac{ax^2+b}{x}\, dx = 18 + 5\ln 2,$$

find the values of the rational constants a and b.

8



The diagram shows the curve with equation $y = 6 - 2e^x$.

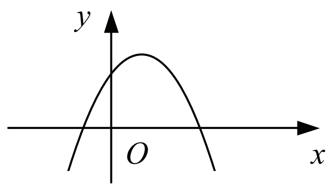
a Find the coordinates of the point P where the curve crosses the x-axis.

b Show that the area of the region enclosed by the curve and the coordinate axes is $6 \ln 3 - 4$.

9 Using the substitution $u = \cot x$, show that

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot^2 x \operatorname{cosec}^4 x \; dx = \tfrac{2}{15}(21\sqrt{3} - 4).$$

10



The diagram shows the curve with parametric equations

$$x = t + 1, \quad y = 4 - t^2.$$

a Show that the area of the region bounded by the curve and the x-axis is given by

$$\int_{-2}^{2} (4 - t^2) \; dt.$$

b Hence, find the area of this region.

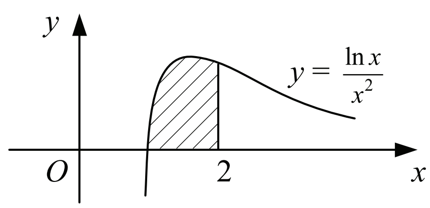
11 **a** Given that k is a constant, show that

$$\frac{d}{dx}(x^2 \sin 2x + 2kx \cos 2x - k \sin 2x) = 2x^2 \cos 2x + (2 - 4k)x \sin 2x.$$

b Using your answer to part **a** with a suitable value of k, or otherwise, find

$$\int x^2 \cos 2x \; dx.$$

12



The shaded region in the diagram is bounded by the curve with equation $y = \frac{\ln x}{x^2}$, the x-axis and the line $x = 2$. Use integration by parts to show that the area of the shaded region is $\frac{1}{2}(1 - \ln 2)$.

13

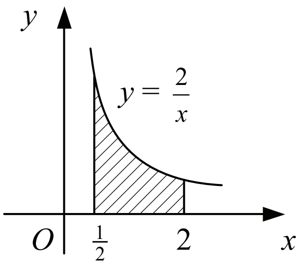
$$f(x) \equiv \frac{x + 16}{3x^3 + 11x^2 + 8x - 4}$$

a Factorise completely $3x^3 + 11x^2 + 8x - 4$.

b Express $f(x)$ in partial fractions.

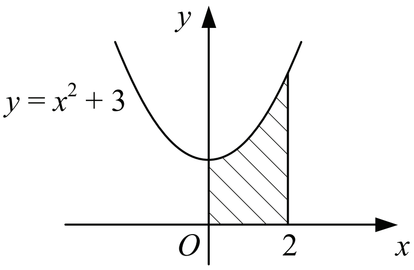
c Show that $\int_{-1}^{0} f(x) \; dx = -(1 + 3 \ln 2)$.

1



The shaded region in the diagram is bounded by the curve $y = \frac{2}{x}$, the x-axis and the lines $x = \frac{1}{2}$ and $x = 2$. Show that when the shaded region is rotated through 360° about the x-axis, the volume of the solid formed is 6π.

2



The shaded region in the diagram, bounded by the curve $y = x^2 + 3$, the coordinate axes and the line $x = 2$, is rotated through 2π radians about the x-axis.

Show that the volume of the solid formed is approximately 127.

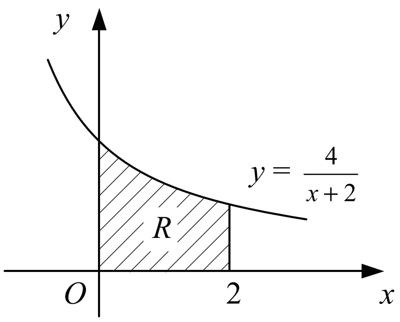
3 The region enclosed by the given curve, the x-axis and the given ordinates is rotated through 360° about the x-axis. Find the exact volume of the solid formed in each case.

a $y = 2e^{\frac{x}{2}}$, $x = 0$, $x = 1$ **b** $y = \frac{3}{x^2}$, $x = -2$, $x = -1$

c $y = 1 + \frac{1}{x}$, $x = 3$, $x = 9$ **d** $y = \frac{3x^2 + 1}{x}$, $x = 1$, $x = 2$

e $y = \frac{1}{\sqrt{x+2}}$, $x = 2$, $x = 6$ **f** $y = e^{1-x}$, $x = -1$, $x = 1$

4



The diagram shows part of the curve with equation $y = \frac{4}{x+2}$.

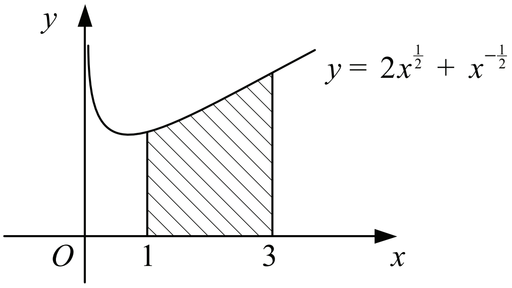
The shaded region, R, is bounded by the curve, the coordinate axes and the line $x = 2$.

a Find the area of R, giving your answer in the form $k \ln 2$.

The region R is rotated through 2π radians about the x-axis.

b Show that the volume of the solid formed is 4π.

5



The diagram shows the curve with equation $y = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}$.

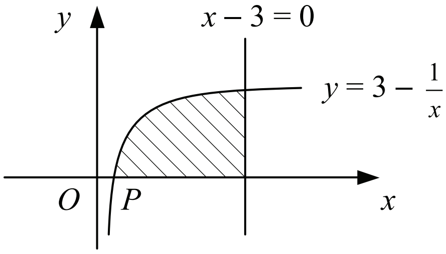
The shaded region bounded by the curve, the x-axis and the lines $x = 1$ and $x = 3$ is rotated through 2π radians about the x-axis. Find the volume of the solid generated, giving your answer in the form $\pi(a + \ln b)$ where a and b are integers.

6 **a** Sketch the curve $y = 3x - x^2$, showing the coordinates of any points where the curve intersects the coordinate axes.

The region bounded by the curve and the x-axis is rotated through 360° about the x-axis.

b Show that the volume of the solid generated is $\frac{81}{10}\pi$.

7



The diagram shows the curve with equation $y = 3 - \frac{1}{x}$, $x > 0$.

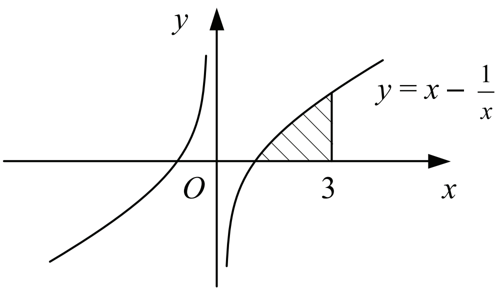
a Find the coordinates of the point P where the curve crosses the x-axis.

The shaded region is bounded by the curve, the straight line $x - 3 = 0$ and the x-axis.

b Find the area of the shaded region.

c Find the volume of the solid formed when the shaded region is rotated completely about the x-axis, giving your answer in the form $\pi(a + b \ln 3)$ where a and b are rational.

8



The diagram shows the curve $y = x - \frac{1}{x}$, $x \neq 0$.

a Find the coordinates of the points where the curve crosses the x-axis.

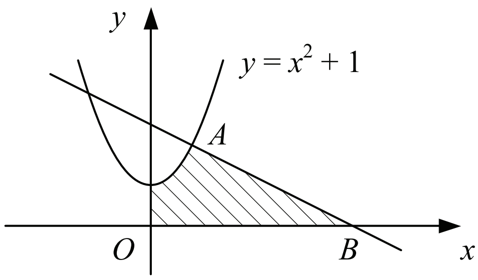
The shaded region is bounded by the curve, the x-axis and the line $x = 3$.

b Show that the area of the shaded region is $4 - \ln 3$.

The shaded region is rotated through 360° about the x-axis.

c Find the volume of the solid generated as an exact multiple of π.

1



The diagram shows the curve $y = x^2 + 1$ which passes through the point A (1, 2).

a Find an equation of the normal to the curve at the point A.

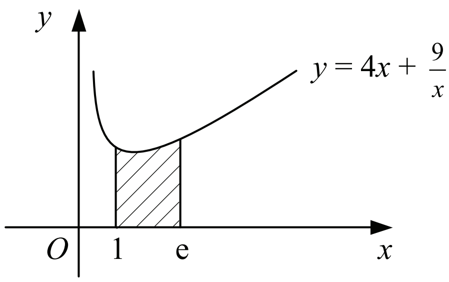
The normal to the curve at A meets the x-axis at the point B as shown.

b Find the coordinates of B.

The shaded region bounded by the curve, the coordinate axes and the line AB is rotated through 2π radians about the x-axis.

c Show that the volume of the solid formed is $\frac{36}{5}\pi$.

2



The shaded region in the diagram is bounded by the curve with equation $y = 4x + \frac{9}{x}$, the x-axis and the lines $x = 1$ and $x = \mathrm{e}$.

a Find the area of the shaded region, giving your answer in terms of e.

b Find, to 3 significant figures, the volume of the solid formed when the shaded region is rotated completely about the x-axis.

3 The region enclosed by the given curve, the x-axis and the given ordinates is rotated through 2π radians about the x-axis. Find the exact volume of the solid formed in each case.

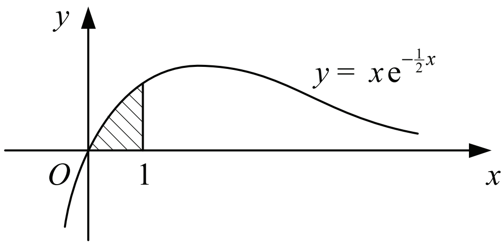
a $y = \operatorname{cosec} x,\quad x = \frac{\pi}{6},\quad x = \frac{\pi}{3}$

b $y = \sqrt{\frac{x+3}{x+2}},\quad x = 1,\quad x = 4$

c $y = 1 + \cos 2x,\quad x = 0,\quad x = \frac{\pi}{4}$

d $y = x^{\frac{1}{2}}\mathrm{e}^{2-x},\quad x = 1,\quad x = 2$

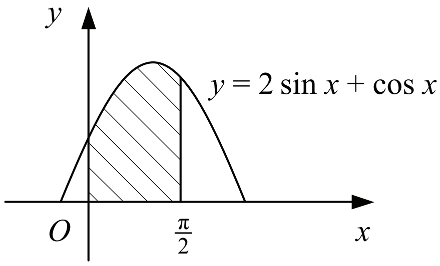
4



The shaded region in the diagram, bounded by the curve $y = x\mathrm{e}^{-\frac{1}{2}x}$, the x-axis and the line $x = 1$, is rotated through 360° about the x-axis.

Show that the volume of the solid formed is $\pi(2 - 5\mathrm{e}^{-1})$.

5



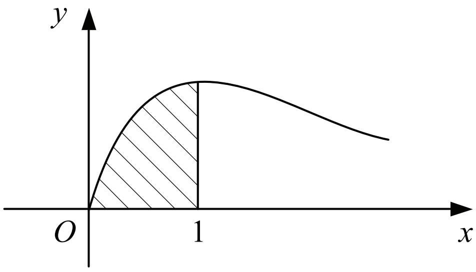
The diagram shows part of the curve with equation $y = 2 \sin x + \cos x$.

The shaded region is bounded by the curve in the interval $0 \leq x < \frac{\pi}{2}$, the positive coordinate axes and the line $x = \frac{\pi}{2}$.

a Find the area of the shaded region.

b Show that the volume of the solid formed when the shaded region is rotated through 2π radians about the x-axis is $\frac{1}{4}\pi(5\pi + 8)$.

6



The diagram shows part of the curve with parametric equations

$$x = \tan \theta, \quad y = \sin 2\theta, \quad 0 \leq \theta < \frac{\pi}{2}.$$

The shaded region is bounded by the curve, the x-axis and the line $x = 1$.

a Write down the value of the parameter θ at the points where $x = 0$ and where $x = 1$.

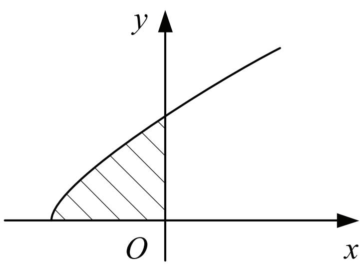
The shaded region is rotated through 2π radians about the x-axis.

b Show that the volume of the solid formed is given by

$$4\pi \int_0^{\frac{\pi}{4}} \sin^2 \theta \ \mathrm{d}\theta.$$

c Evaluate this integral.

7



The diagram shows part of the curve with parametric equations

$$x = t^2 - 1, \quad y = t(t + 1), \quad t \geq 0.$$

a Find the value of the parameter t at the points where the curve meets the coordinate axes.

The shaded region bounded by the curve and the coordinate axes is rotated through 2π radians about the x-axis.

b Find the volume of the solid formed, giving your answer in terms of π.

1 Find the general solution of each differential equation.

a $\frac{dy}{dx} = (x+2)^3$ **b** $\frac{dy}{dx} = 4\cos 2x$ **c** $\frac{dx}{dt} = 3e^{2t} + 2$

d $(2-x)\frac{dy}{dx} = 1$ **e** $\frac{dN}{dt} = t\sqrt{t^2+1}$ **f** $\frac{dy}{dx} = xe^x$

2 Find the particular solution of each differential equation.

a $\frac{dy}{dx} = e^{-x}$, $y = 3$ when $x = 0$ **b** $\frac{dy}{dt} = \tan^3 t \sec^2 t$, $y = 1$ when $t = \frac{\pi}{3}$

c $(x^2-3)\frac{du}{dx} = 4x$, $u = 5$ when $x = 2$ **d** $\frac{dy}{dx} = 3\cos^2 x$, $y = \pi$ when $x = \frac{\pi}{2}$

3 **a** Express $\frac{x-8}{x^2-x-6}$ in partial fractions.

b Given that

$$(x^2-x-6)\frac{dy}{dx} = x-8,$$

and that $y = \ln 9$ when $x = 1$, show that when $x = 2$, the value of y is ln 32.

4 Find the general solution of each differential equation.

a $\frac{dy}{dx} = 2y + 3$ **b** $\frac{dy}{dx} = \sin^2 2y$ **c** $\frac{dy}{dx} = xy$

d $(x+1)\frac{dy}{dx} = y$ **e** $\frac{dy}{dx} = \frac{x^2-2}{y}$ **f** $\frac{dy}{dx} = 2\cos x \cos^2 y$

g $\sqrt{x}\frac{dy}{dx} = e^{y-3}$ **h** $y\frac{dy}{dx} = xy^2 + 3x$ **i** $\frac{dy}{dx} = xy\sin x$

j $\frac{dy}{dx} = e^{2x-y}$ **k** $(y-3)\frac{dy}{dx} = xy(y-1)$ **l** $\frac{dy}{dx} = y^2 \ln x$

5 Find the particular solution of each differential equation.

a $\frac{dy}{dx} = \frac{x}{2y}$, $y = 3$ when $x = 4$ **b** $\frac{dy}{dx} = (y+1)^3$, $y = 0$ when $x = 2$

c $(\tan^2 x)\frac{dy}{dx} = y$, $y = 1$ when $x = \frac{\pi}{2}$ **d** $\frac{dy}{dx} = \frac{y+2}{x-1}$, $y = 6$ when $x = 3$

e $\frac{dy}{dx} = x^2 \tan y$, $y = \frac{\pi}{6}$ when $x = 0$ **f** $\frac{dy}{dx} = \sqrt{\frac{y}{x+3}}$, $y = 16$ when $x = 1$

g $e^x\frac{dy}{dx} = x\,\text{cosec}\, y$, $y = \pi$ when $x = -1$ **h** $\frac{dy}{dx} = \frac{1+\cos y}{2x^2 \sin y}$, $y = \frac{\pi}{3}$ when $x = 1$

1 **a** Express $\frac{x+4}{(1+x)(2-x)}$ in partial fractions.

b Given that $y = 2$ when $x = 3$, solve the differential equation

$$\frac{dy}{dx} = \frac{y(x+4)}{(1+x)(2-x)}.$$

2 Given that $y = 0$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = e^{x+y} \cos x.$$

3 Given that $\frac{dy}{dx}$ is inversely proportional to x and that $y = 4$ and $\frac{dy}{dx} = \frac{5}{3}$ when $x = 3$, find an expression for y in terms of x.

4 A quantity has the value N at time t hours and is increasing at a rate proportional to N.

a Write down a differential equation relating N and t.

b By solving your differential equation, show that

$$N = Ae^{kt},$$

where A and k are constants and k is positive.

Given that when $t = 0$, $N = 40$ and that when $t = 5$, $N = 60$,

c find the values of A and k,

d find the value of N when $t = 12$.

5 A cube is increasing in size and has volume V cm^3 and surface area A cm^2 at time t seconds.

a Show that

$$\frac{dV}{dA} = k\sqrt{A},$$

where k is a positive constant.

Given that the rate at which the volume of the cube is increasing is proportional to its surface area and that when $t = 10$, $A = 100$ and $\frac{dA}{dt} = 5$,

b show that

$$A = \tfrac{1}{16}(t + 30)^2.$$

6 At time $t = 0$, a piece of radioactive material has mass 24 g. Its mass after t days is m grams and is decreasing at a rate proportional to m.

a By forming and solving a suitable differential equation, show that

$$m = 24e^{-kt},$$

where k is a positive constant.

After 20 days, the mass of the material is found to be 22.6 g.

b Find the value of k.

c Find the rate at which the mass is decreasing after 20 days.

d Find how long it takes for the mass of the material to be halved.

7 A quantity has the value P at time t seconds and is decreasing at a rate proportional to $\sqrt{P}$.

a By forming and solving a suitable differential equation, show that

$$P = (a - bt)^2,$$

where a and b are constants.

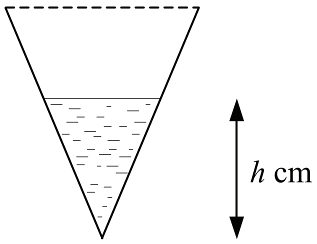
Given that when $t = 0$, $P = 400$,

b find the value of a.

Given also that when $t = 30$, $P = 100$,

c find the value of P when $t = 50$.

8



The diagram shows a container in the shape of a right-circular cone. A quantity of water is poured into the container but this then leaks out from a small hole at its vertex.

In a model of the situation it is assumed that the rate at which the volume of water in the container, V cm^3, decreases is proportional to V. Given that the depth of the water is h cm at time t minutes,

a show that

$$\frac{dh}{dt} = -kh,$$

where k is a positive constant.

Given also that $h = 12$ when $t = 0$ and that $h = 10$ when $t = 20$,

b show that

$$h = 12e^{-kt},$$

and find the value of k,

c find the value of t when $h = 6$.

9 **a** Express $\frac{1}{(1+x)(1-x)}$ in partial fractions.

In an industrial process, the mass of a chemical, m kg, produced after t hours is modelled by the differential equation

$$\frac{dm}{dt} = ke^{-t}(1 + m)(1 - m),$$

where k is a positive constant.

Given that when $t = 0$, $m = 0$ and that the initial rate at which the chemical is produced is 0.5 kg per hour,

b find the value of k,

c show that, for $0 \le m < 1$, $\ln\left(\frac{1+m}{1-m}\right) = 1 - e^{-t}$.

d find the time taken to produce 0.1 kg of the chemical,

e show that however long the process is allowed to run, the maximum amount of the chemical that will be produced is about 462 g.

1 Use the trapezium rule with n intervals of equal width to estimate the value of each integral.

a $\int_1^5 x \ln(x+1)\ dx \quad n = 2$

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\ dx \quad n = 2$

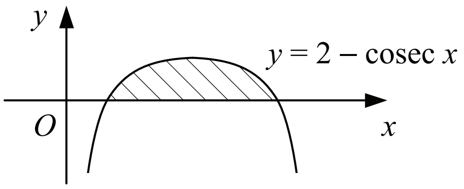
c $\int_{-2}^{2} e^{\frac{x^2}{10}}\ dx \quad n = 4$

d $\int_0^1 \arccos(x^2 - 1)\ dx \quad n = 4$

e $\int_0^{0.5} \sec^2(2x - 1)\ dx \quad n = 5$

f $\int_0^6 x^3 e^{-x}\ dx \quad n = 6$

2



The diagram shows the curve with equation $y = 2 - \operatorname{cosec} x$, $0 < x < \pi$.

a Find the exact x-coordinates of the points where the curve crosses the x-axis.

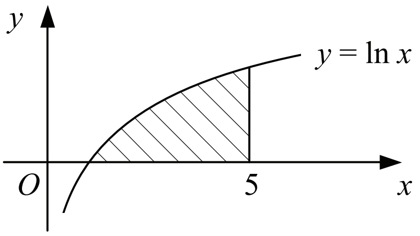
b Use the trapezium rule with four intervals of equal width to estimate the area of the shaded region bounded by the curve and the x-axis.

3 $$f(x) \equiv \tfrac{\pi}{6} + \arcsin(\tfrac{1}{2}x), \quad x \in \mathbb{R}, -2 \le x \le 2.$$

a Use the trapezium rule with three strips to estimate the value of the integral $I = \int_{-1}^{2} f(x)\ dx$.

b Use the trapezium rule with six strips to find an improved estimate for I.

4



The shaded region in the diagram is bounded by the curve $y = \ln x$, the x-axis and the line $x = 5$.

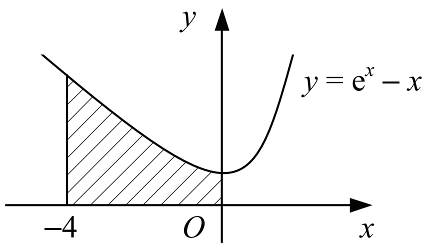
a Estimate the area of the shaded region to 3 decimal places using the trapezium rule with

i 2 strips **ii** 4 strips **iii** 8 strips

b By considering your answers to part **a**, suggest a more accurate value for the area of the shaded region correct to 3 decimal places.

c Use integration to find the true value of the area correct to 3 decimal places.

5



The shaded region in the diagram is bounded by the curve $y = e^x - x$, the coordinate axes and the line $x = -4$. Use the trapezium rule with five equally-spaced ordinates to estimate the volume of the solid formed when the shaded region is rotated completely about the x-axis.