目录

IAL FP3 Mark Scheme

Question Number	Scheme	Notes	Marks
1(a)	$4\sinh^3 x + 3\sinh x = 4\left(\frac{e^x - e^{-x}}{2}\right)^3 + 3\left(\frac{e^x - e^{-x}}{2}\right)$ $= 4\left(\frac{e^{3x} - 3e^x + 3e^{-x} - e^{-3x}}{8}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right)$ Uses $\sinh x = \frac{e^x - e^{-x}}{2}$ on both sinh terms and attempts to cube the bracket (min accepted is a linear x a quadratic bracket)		M1
	$= \frac{1}{2}e^{3x} - \frac{3}{2}e^x + \frac{3}{2}e^{-x} - \frac{1}{2}e^{-3x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x}$ $= \frac{e^{3x} - e^{-3x}}{2} = \sinh 3x$*		A1*
			(2)
(b)	$\sinh 3x = 19\sinh x \Rightarrow 4\sinh^3 x + 3\sinh x = 19\sinh x$ $\Rightarrow 4\sinh^3 x - 16\sinh x = 0$ Uses the result from (a) and combines terms		M1
	$(\sinh x = 0 \text{ or}) \ \sinh^2 x = 4$	$\sinh^2 x = 4$ or $\sinh x = (\pm)2$	A1
	(0, 0)	States the origin as one intersection	B1
	$\ln\left(2+\sqrt{5}\right)$ **and** $-\ln\left(2+\sqrt{5}\right)$	Two correct non-zero x values(allow e.g. $\ln\left(-2+\sqrt{5}\right)$ for $-\ln\left(2+\sqrt{5}\right)$)	A1
	$\left(\ln\left(2+\sqrt{5}\right), 38\right)$ **and** $\left(-\ln\left(2+\sqrt{5}\right), -38\right)$	Two correct **points** (allow e.g. $\ln\left(-2+\sqrt{5}\right)$ for $-\ln\left(2+\sqrt{5}\right)$)	A1
			(5)
	Alternative for (b) using exponentials		
	$\sinh 3x = 19\sinh x \Rightarrow \frac{e^{3x} - e^{-3x}}{2} = \frac{19\left(e^x - e^{-x}\right)}{2} \Rightarrow \ldots$ Substitutes the correct exponential forms and collects terms to one side		M1
	$\Rightarrow e^{6x} - 19e^{4x} + 19e^{2x} - 1 = 0$	Correct equation (or equivalent)	A1
	(0, 0)	States the origin as one intersection	B1
	$\frac{1}{2}\ln\left(9+4\sqrt{5}\right)$ **or** $\frac{1}{2}\ln\left(9-4\sqrt{5}\right)$	Two correct non-zero x values (oe)	A1
	$\left(\frac{1}{2}\ln\left(9+4\sqrt{5}\right), 38\right)$ **and** $\left(\frac{1}{2}\ln\left(9-4\sqrt{5}\right), -38\right)$	Two correct **points** (oe)	A1
			Total 7

Question Number	Scheme	Notes	Marks
2(i)	$3x^2+12x+24=3(x^2+4x+8)$ $=3((x+2)^2+4)$	Obtains $3((x+2)^2+...)$ or $3(x+2)^2+...$ Must include 3 now or later	M1
	$3((x+2)^2+4)$ or $3(x+2)^2+12$		A1
	$\int \frac{1}{3x^2+12x+24}\,dx=\frac{1}{3}\int \frac{1}{(x+2)^2+4}\,dx=\frac{1}{6}\arctan\frac{x+2}{2}(+c)$ M1: Use of arctan A1: Fully correct expression (condone omission of + c)		M1A1
			(4)
(ii)	$27-6x-x^2=-(x^2+6x-27)$ $=-((x+3)^2-36)$	Obtains $-((x+3)^2+...)$ or $-(x+3)^2+...$	M1
	$-((x+3)^2-36)$ or $36-(x+3)^2$		A1
	$\int \frac{1}{\sqrt{27-6x-x^2}}\,dx=\int \frac{1}{\sqrt{36-(x+3)^2}}\,dx=\arcsin\left(\frac{x+3}{6}\right)(+c)$ (Or $=-\arccos\left(\frac{x+3}{6}\right)(+c)$) M1: Use of arcsin (or – arccos) A1: Fully correct expression (condone omission of + c)		M1A1
			(4)
			Total 8

Question Number	Scheme	Notes	Marks
3	$\mathbf{M}=\begin{pmatrix}3 & -4 & k\\ 1 & -2 & k\\ 1 & -5 & 5\end{pmatrix}$		
(a)	$\lvert\mathbf{M}-\lambda\mathbf{I}\rvert=\mathbf{M}=\begin{vmatrix}3-\lambda & -4 & k\\ 1 & -2-\lambda & k\\ 1 & -5 & 5-\lambda\end{vmatrix}=\begin{vmatrix}0 & -4 & k\\ 1 & -5 & k\\ 1 & -5 & 2\end{vmatrix}$ $(0)+4[2-k]+k[-5+5]$ Attempts $\lvert\mathbf{M}-\lambda\mathbf{I}\rvert$ using $\lambda=3$		M1
	$(0)+4[2-k]+k[-5+5]=0\Rightarrow k=...$ Uses $\lvert\mathbf{M}-\lambda\mathbf{I}\rvert=0$ and solves for k		M1
	$k=2$	Cao	A1
			(3)
(b)	$(3-\lambda)[(\lambda+2)(\lambda-5)+10]+4(5-\lambda-2)+2(-5+2+\lambda)=0$ Attempts $\lvert\mathbf{M}-\lambda\mathbf{I}\rvert=0$ using their value of k		M1
	$\Rightarrow(3-\lambda)[(\lambda+2)(\lambda-5)+12]=0$ $(\lambda+2)(\lambda-5)+12\Rightarrow\lambda^2-3\lambda+2=0\Rightarrow(\lambda-2)(\lambda-1)=0\Rightarrow\lambda=...$ Uses $\lambda=3$as a factor to obtain and solve a 3TQ to find the other eigenvalues (Alternatively may use calculator to solve $\lambda^3-6\lambda^2+11\lambda-6=0$)		M1
	$\lambda=1,\ 2$	Correct values	A1
			(3)
(c)	$\begin{pmatrix}3 & -4 & 2\\ 1 & -2 & 2\\ 1 & -5 & 5\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}\Rightarrow\begin{matrix}3x-4y+2z=3x\\ x-2y+2z=3y\\ x-5y+5z=3z\end{matrix}$	Uses the eigenvalue 3 and their k to form at least 2 equations in x, y and z	M1
	$\alpha\begin{pmatrix}1\\ 1\\ 2\end{pmatrix}$ (α a constant)	Any correct eigenvector. Allow any constant multiple of **i** + **j** + 2**k**	A1
	$\frac{1}{\sqrt{6}}\begin{pmatrix}1\\ 1\\ 2\end{pmatrix}$	Correct normalised vector	A1
			(3)
			Total 9

Question Number	Scheme	Notes	Marks
4.	$I_n = \int x^n \cos x \, dx$		
(a)	$\int x^n \cos x \, dx = x^n \sin x - \int n x^{n-1} \sin x \, dx$ M1: Parts in the correct direction A1: Correct expression		M1A1
	$= x^n \sin x - \left\{ -n x^{n-1} \cos x + \int n(n-1) x^{n-2} \cos x \, dx \right\}$ Uses integration by parts again (**dependent on the first M**)		dM1
	$= x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}$ * Fully correct proof with no errors		A1*
			(4)
ALT			
	$I_n = \int x^n \cos x \, dx = \int x^{n-1} (x \cos x) \, dx$		
	$= x^n \sin x + x^{n-1} \cos x - (n-1) \int x^{n-2} (x \sin x + \cos x) \, dx$ M1: Parts in the correct direction A1: Correct expression		M1A1
	$= x^n \sin x + x^{n-1} \cos x - (n-1) \int x^{n-1} \sin x dx - (n-1) I_{n-2}$		
	$= x^n \sin x + x^{n-1} \cos x - (n-1)\left\{ -x^{n-1} \cos x + (n-1) I_{n-2} \right\} - (n-1) I_{n-2}$ Uses integration by parts again (**dependent on the first M**)		dM1
	$= x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}$ * Fully correct proof with no errors		A1*
(b)	$I_0 = \sin x \ \ (+k)$		B1
	$I_4 = x^4 \sin x + 4x^3 \cos x - 12 I_2$	Applies the reduction formula once for I_4 or I_2	M1
	$= x^4 \sin x + 4x^3 \cos x - 12\left(x^2 \sin x + 2x \cos x - 2 I_0 \right)$ Applies the reduction formula again and obtains an expression for I_4 which can include I_0 but not I_2		M1
	$= \left(x^4 - 12x^2 + 24 \right) \sin x + \left(4x^3 - 24x \right) \cos x + c$ Award A1 for either bracket and A1 for the other If the answer is not factorised but is otherwise correct, award A1A0		A1A1
			(5)
			Total 9

Question Number	Scheme	Notes	Marks
5	$\frac{x^2}{25}-\frac{y^2}{4}=1 \quad y=mx+c$		
(a)	$\frac{x^2}{25}-\frac{(mx+c)^2}{4}=1\Rightarrow 4x^2-25\left(m^2x^2+2cmx+c^2\right)=100$ Substitutes to obtain a quadratic in x and eliminates fractions		M1
	$4x^2-25\left(m^2x^2+2cmx+c^2\right)=100$ $\left(\Rightarrow\left(25m^2-4\right)x^2+50cmx+25c^2+100=0\right)$ Correct 3TQ		A1
	$"b^2=4ac"\Rightarrow(50cm)^2=4\left(25m^2-4\right)\left(25c^2+100\right)$ Uses '$b^2=4ac$' or equivalent		M1
	$2500c^2m^2=2500c^2m^2+10000m^2-400c^2-1600$ $10000m^2=400c^2+1600$ $25m^2=c^2+4$ * Fully correct proof with no errors		A1*
			(4)
ALT 1	Using hyperbolic parameters:		
	$x=5\cosh t, y=2\sinh t\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\cosh t}{5\sinh t}$		
	$\frac{2\cosh t}{5\sinh t}(x-5\cosh t)=y-2\sinh t$ M1: Attempts the equation of the tangent A1: Correct equation (no simplification needed)		M1A1
	$y=\frac{2\cosh t}{5\sinh t}x-\frac{2\cosh^2 t-25\sinh^2 t}{\sinh t}$		
	$25m^2=\frac{4\cosh^2 t}{\sinh^2 t},\ 4+c^2=4+\frac{4}{\sinh^2 t}=\frac{4\left(\sinh^2 t+1\right)}{\sinh^2 t}=\frac{4\cosh^2 t}{\sinh^2 t}$ Extracts $25m^2$ and $4+c^2$ from their equation		M1
	$\therefore 25m^2=4+c^2$ * Fully correct proof with no errors		A1*
			(4)
ALT 2	Using trigonometric parameters:		
	$x=5\sec t, y=2\tan t\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\sec t}{5\tan t}$		
	$\frac{2\sec t}{5\tan t}(x-5\sec t)=y-2\tan t$ M1: Attempts the equation of the tangent A1: Correct equation (no simplification needed)		M1A1
	$y=\frac{2\sec t}{5\tan t}x+\frac{2\tan^2 t-2\sec^2 t}{\tan t}$		
	$25m^2=\frac{4\sec^2 t}{\tan^2 t}=\frac{4}{\sin^2 t}\quad 4+c^2=4\left(1+\frac{1}{\tan^2 t}\right)=4\left(\frac{\sin^2 t+\cos^2 t}{\sin^2 t}\right)=\frac{4}{\sin^2 t}$ Extracts $25m^2$ and $4+c^2$ from their equation		M1
	$\therefore 25m^2=4+c^2$ * Fully correct proof with no errors		A1*
			(4)

(b)	$25m^2 = c^2 + 4$ and $2 = m + c$ $25m^2 = (2-m)^2 + 4$ or $25(2-c)^2 = c^2 + 4$ Uses the given hyperbola and the straight line with the result from (a) to obtain an equation in m or c		M1
	$24m^2 + 4m - 8 = 0$ or $24c^2 - 100c + 96 = 0$	Correct 3TQ in m or c	A1
	$24m^2 + 4m - 8 = 0 \Rightarrow m = \frac{1}{2}, -\frac{2}{3}$ Or $24c^2 - 100c + 96 = 0 \Rightarrow c = \frac{3}{2}, \frac{8}{3}$	Solves their 3TQ in m or c	M1
	$y = \frac{1}{2}x + \frac{3}{2}$ **or** $y = -\frac{2}{3}x + \frac{8}{3}$	One correct tangent	A1
	$y = \frac{1}{2}x + \frac{3}{2}$ **and** $y = -\frac{2}{3}x + \frac{8}{3}$	Both correct tangents	A1
			(5)
(c)	$m = \frac{1}{2}, c = \frac{3}{2} \Rightarrow \frac{9}{4}x^2 + \frac{75}{2}x + \frac{625}{4} = 0 \Rightarrow x = \ldots$ **or** $m = -\frac{2}{3}, c = \frac{8}{3} \Rightarrow \frac{64}{9}x^2 - \frac{800}{9}x + \frac{2500}{9} = 0 \Rightarrow x = \ldots$ Uses one of their m and c pairs and solves for x		M1
	$x = -\frac{25}{3}, y = -\frac{8}{3}$ **or** $x = \frac{25}{4}, y = -\frac{3}{2}$	One correct point	A1
	$x = -\frac{25}{3}, y = -\frac{8}{3}$ **and** $x = \frac{25}{4}, y = -\frac{3}{2}$	Both correct points	A1
			(3)
			Total 12

Question Number	Scheme	Notes	Marks
6(a)	$\mathbf{A}=\begin{pmatrix}1&-1&1\\1&1&1\\1&2&a\end{pmatrix}$		
	$\lvert\mathbf{A}\rvert=a-2+a-1+2-1(=2a-2)$	Correct determinant in any form	B1
	$\begin{pmatrix}1&-1&1\\1&1&1\\1&2&a\end{pmatrix}\to\begin{pmatrix}a-2&a-1&1\\-a-2&a-1&3\\-2&0&2\end{pmatrix}\to\begin{pmatrix}a-2&1-a&1\\a+2&a-1&-3\\-2&0&2\end{pmatrix}$ Applies the correct method to reach at least a matrix of cofactors 2 correct rows or 2 correct columns needed		M1
	$\begin{pmatrix}a-2&1-a&1\\a+2&a-1&-3\\-2&0&2\end{pmatrix}\to\begin{pmatrix}a-2&a+2&-2\\1-a&a-1&0\\1&-3&2\end{pmatrix}$ Correct transpose of cofactors		A1
	$\mathbf{A}^{-1}=\dfrac{1}{2a-2}\begin{pmatrix}a-2&a+2&-2\\1-a&a-1&0\\1&-3&2\end{pmatrix}$	Correct inverse	A1
			(4)
(b)	$a=4\Rightarrow\mathbf{A}^{-1}=\dfrac{1}{6}\begin{pmatrix}2&6&-2\\-3&3&0\\1&-3&2\end{pmatrix}$	Correct inverse (follow through their matrix from (a))	B1ft
	$=\dfrac{1}{6}\begin{pmatrix}2&6&-2\\-3&3&0\\1&-3&2\end{pmatrix}\begin{pmatrix}12-6\lambda\\4+2\lambda\\6+3\lambda\end{pmatrix}=\ldots$	Attempt to multiply the parametric form of l_2 by their inverse	M1
	$=\begin{pmatrix}6-\lambda\\-4+4\lambda\\2-\lambda\end{pmatrix}$	Correct parametric form	A1
	$\mathbf{r}=\begin{pmatrix}6\\-4\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\-1\end{pmatrix}$	Correct equation (allow equivalent forms) but if given as $l=\ldots$ award A0	A1
			(4)
			Total 8

	Alternatives for (b)		
(i)	$a=4 \Rightarrow \mathbf{A}^{-1}=\frac{1}{6}\begin{pmatrix}2 & 6 & -2\\ -3 & 3 & 0\\ 1 & -3 & 2\end{pmatrix}$	Correct inverse (follow through their matrix from (a))	B1ft
	$\frac{1}{6}\begin{pmatrix}2 & 6 & -2\\ -3 & 3 & 0\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}12\\ 4\\ 6\end{pmatrix}=\frac{1}{6}\begin{pmatrix}36\\ -24\\ 12\end{pmatrix}$ $\frac{1}{6}\begin{pmatrix}2 & 6 & -2\\ -3 & 3 & 0\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}-6\\ 2\\ 3\end{pmatrix}=\frac{1}{6}\begin{pmatrix}-6\\ 24\\ -6\end{pmatrix}$	Attempt $\mathbf{A}^{-1}$(point on l_2) **and** $\mathbf{A}^{-1}$(direction of l_2) **Both** correct (NB No ft)	M1 A1
	$\mathbf{r}=\begin{pmatrix}6\\ -4\\ 2\end{pmatrix}+\lambda\begin{pmatrix}-1\\ 4\\ -1\end{pmatrix}$	Correct equation (allow equivalent forms) but if given as $l=$ ... award A0	A1
			(4)
(ii)	$a=4 \Rightarrow \mathbf{A}^{-1}=\frac{1}{6}\begin{pmatrix}2 & 6 & -2\\ -3 & 3 & 0\\ 1 & -3 & 2\end{pmatrix}$	Correct inverse (follow through their matrix from (a))	B1ft
	$\frac{1}{6}\begin{pmatrix}2 & 6 & -2\\ -3 & 3 & 0\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}12\\ 4\\ 6\end{pmatrix}=\frac{1}{6}\begin{pmatrix}36\\ -24\\ 12\end{pmatrix}$ $\frac{1}{6}\begin{pmatrix}2 & 6 & -2\\ -3 & 3 & 0\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}6\\ 6\\ 9\end{pmatrix}=\frac{1}{6}\begin{pmatrix}30\\ 0\\ 6\end{pmatrix}$	Attempt $\mathbf{A}^{-1}$(point on l_2) for 2 points **Both** correct (NB No ft)	M1 A1
	$\mathbf{r}=\begin{pmatrix}6\\ -4\\ 2\end{pmatrix}+\lambda\begin{pmatrix}-1\\ 4\\ -1\end{pmatrix}$	Obtain the direction vector and deduce correct equation (allow equivalent forms) but if given as $l=$ ... award A0	A1
			(4)

Question Number	Scheme	Notes	Marks
7	$x=\cosh t+t,\quad y=\cosh t-t$		
(a)	$\frac{dx}{dt}=\sinh t+1,\ \frac{dy}{dt}=\sinh t-1$	Correct derivatives	B1
	$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\sinh^2 t+2\sinh t+1+\sinh^2 t-2\sinh t+1$ $=2\sinh^2 t+2$ M1: Squares correctly, cancels and collects terms		M1
	$=2\left(1+\sinh^2 t\right)=2\cosh^2 t$ *	Uses $\cosh^2 t=1+\sinh^2 t$ to complete the proof with no errors	A1*
			(3)
(b)	$S=2\pi\int y\,ds=2\pi\int(\cosh t-t)\sqrt{2}\cosh t\,dt$	Uses $S=2\pi\int y\,ds$ with the given y and the result from part (a)	M1
	$=2\sqrt{2}\pi\int_0^{\ln 3}\left(\cosh^2 t-t\cosh t\right)dt$ *	Correct proof with no errors	A1*
			(2)
(c)	$\int\cosh^2 t\,dt=\int\pm\frac{1}{2}\pm\frac{1}{2}\cosh 2t\,dt$	Uses $\cosh^2 t=\pm\frac{1}{2}\pm\frac{1}{2}\cosh 2t$	M1
	$\int t\cosh t\,dt=t\sinh t-\int\sinh t\,dt$	Attempts integration by parts the right way round on $t\cosh t$	M1
		Correct expression	A1
	$S=\left(2\sqrt{2}\pi\right)\int\left(\cosh^2 t-t\cosh t\right)dt=\left(2\sqrt{2}\pi\right)\left[\frac{1}{2}t+\frac{1}{4}\sinh 2t-t\sinh t+\cosh t\right]$ A1: 2 correct terms A1: All correct		A1A1
	$(S=)2\sqrt{2}\pi\left\{\left(\frac{1}{2}\ln 3+\frac{10}{9}-\frac{4}{3}\ln 3+\frac{5}{3}\right)-(1)\right\}$ dM1: Correct use of limits 0 and ln3 depends on both preceding M marks		dM1
	$S=\frac{1}{9}\sqrt{2}\pi\left(32-15\ln 3\right)$	cao	A1 (7)
			Total 12
	Alternative for (c)		
	$\int\cosh^2 t\,dt=\int\left(\frac{e^t+e^{-t}}{2}\right)^2dt$ $=\frac{1}{4}\int\left(e^{2t}+2+e^{-2t}\right)dt$	Substitutes the exponential form and attempts to square	M1
	$\int t\cosh t\,dt=\frac{1}{2}\int t\left(e^t+e^{-t}\right)dt$ $=\frac{1}{2}te^t-\frac{1}{2}\int te^t dt-\left\{\frac{1}{2}te^{-t}-\frac{1}{2}\int e^{-t}dt\right\}$	Substitutes the exponential form and attempts integration by parts the right way round Correct expression	M1 A1
	$(S=)\left(2\sqrt{2}\pi\right)\left\{\frac{1}{4}\left(\frac{1}{2}e^{2t}+2t-\frac{1}{2}e^{-2t}\right)-\frac{1}{2}te^t+\frac{1}{2}e^t+\frac{1}{2}te^{-t}-\frac{1}{2}e^{-t}\right\}$ A1: either integral correct A1: other integral correct but both must be in a complete expression for S		A1A1
	Depends on both M marks above	Correct use of limits 0 and ln3	dM1
	$S=\frac{1}{9}\sqrt{2}\pi\left(32-15\ln 3\right)$	cao	A1

	Alternative for the first 3 marks of (c)		
	$=2\sqrt{2}\pi\int\left(\cosh^2 t - t\cosh t\right)\mathrm{d}t$ $=2\sqrt{2}\pi\int \cosh t\left(\cosh t - t\right)\mathrm{d}t$		
	$2\sqrt{2}\pi\left(\left[\sinh t\left(\cosh t - t\right)\right]-\int \sinh t\left(\sinh t - 1\right)\mathrm{d}t\right)$		
	$2\sqrt{2}\pi\left(\left[\sinh t\left(\cosh t - t\right)\right]-\left[\cosh t\left(\sinh t - 1\right)\right]+\int \cosh^2 t\,\mathrm{d}t\right)$ M1 (2^{nd} on e-PEN): Use parts twice A1 Correct expression		M1A1
	$\int \cosh^2 t\,\mathrm{d}t = \int \pm\frac{1}{2}\pm\frac{1}{2}\cosh 2t\,\mathrm{d}t$	Uses $\cosh^2 t = \pm\frac{1}{2}\pm\frac{1}{2}\cosh 2t$	M1 (1st on e-PEN)
	Rest as main scheme		

Question Number	Scheme	Notes	Marks
8(a)	$\mathbf{n}=\begin{pmatrix}1\\-5\\3\end{pmatrix}\times\begin{pmatrix}3\\-2\\2\end{pmatrix}=\begin{pmatrix}-10+6\\-(2-9)\\-2+15\end{pmatrix}$	Attempt vector product between normal vectors	M1
	$=\begin{pmatrix}-4\\7\\13\end{pmatrix}$	Correct vector	A1
	$x=0\Rightarrow -5y+3z=11,\quad -2y+2z=7$ $\Rightarrow y=-\frac{1}{4}, z=\frac{13}{4}$ or $y=0\Rightarrow x+3z=11,\quad 3x+2z=7$ $\Rightarrow x=-\frac{1}{7}, z=\frac{26}{7}$ or $z=0\Rightarrow x-5y=11, 3x-2y=7$ $\Rightarrow x=1, y=-2$	Correct strategy to find a point on l	M1
		Correct position vector of point on l	A1
	$\mathbf{r}=\mathbf{i}-2\mathbf{j}+\lambda(-4\mathbf{i}+7\mathbf{j}+13\mathbf{k})$	Correct equation. (follow through their position and direction vectors but must be "$\mathbf{r}$ =")	A1ft
			(5)
ALT	$x=11+5y-3z$		
	$3x-2y+2z=7\Rightarrow 3(11+5y-3z)-2y+2z=7$ $\Rightarrow y-\frac{7z}{13}=-\frac{26}{13}\quad\left(z=\frac{13y+26}{7}\right)$ Eliminate one variable		M1
	$x=11+5\left(-\frac{26}{13}+\frac{7z}{13}\right)\Rightarrow z=\frac{13-13x}{4}$	Obtain 2 correct expressions for one of the variables	A1
	$\dfrac{x-1}{-\frac{4}{13}}=\dfrac{y+2}{\frac{7}{13}}=z$	M1 Obtain a Cartesian equation for l A1 Correct equation	M1A1
	$\mathbf{r}=(\mathbf{i}-2\mathbf{j})+\lambda\left(-\frac{4}{13}\mathbf{i}+\frac{7}{13}\mathbf{j}+\mathbf{k}\right)$ oe	Deduce a vector equation for l Follow through their Cartesian equation	A1ft
			(5)

(b)	$\begin{pmatrix}3\\2\\1\end{pmatrix}-\begin{pmatrix}2\\0\\3\end{pmatrix}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$	Correct vector joining P to Q	B1
	$\begin{pmatrix}-4\\7\\13\end{pmatrix}\times\begin{pmatrix}1\\2\\-2\end{pmatrix}=\begin{pmatrix}-40\\5\\-15\end{pmatrix}$	Attempt vector product between the direction of l and their $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$	M1
		Correct vector	A1
	$\sin\theta=\dfrac{\lvert-40\mathbf{i}+5\mathbf{j}-15\mathbf{k}\rvert}{\lvert-4\mathbf{i}+7\mathbf{j}+13\mathbf{k}\rvert\lvert\mathbf{i}+2\mathbf{j}-2\mathbf{k}\rvert}$	Angle between PQ and line n	
	$d=\lvert\overrightarrow{PQ}\rvert\sin\theta$		
	$d=\dfrac{\lvert-40\mathbf{i}+5\mathbf{j}-15\mathbf{k}\rvert}{\lvert-4\mathbf{i}+7\mathbf{j}+13\mathbf{k}\rvert}=\dfrac{1}{\sqrt{234}}\sqrt{40^2+5^2+15^2}$	Fully correct method for the distance	M1
	$d=\dfrac{5\sqrt{481}}{39}$	Cao Allow equivalent **exact** forms e.g. $d=\dfrac{5\sqrt{74}}{\sqrt{234}}$	A1
			(5)
ALT 1	$\mathbf{r}_m=\begin{pmatrix}2\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}-\frac{4}{7}\\1\\\frac{13}{7}\end{pmatrix}$ or $\mathbf{r}_n=\begin{pmatrix}3\\2\\1\end{pmatrix}+\mu\begin{pmatrix}-\frac{4}{7}\\1\\\frac{13}{7}\end{pmatrix}$	Vector equation for either line with their direction vector from (a)	B1ft
	$\overrightarrow{OP}=\begin{pmatrix}2\\0\\3\end{pmatrix}$ $\overrightarrow{ON}=\begin{pmatrix}3-\frac{4}{7}\mu\\2+\mu\\1+\frac{13}{7}\mu\end{pmatrix}$ $\overrightarrow{NP}=\begin{pmatrix}-1+\frac{4}{7}\mu\\-2-\mu\\2-\frac{13}{7}\mu\end{pmatrix}$	Uses either P and the parametric form of a point on n OR Q and the parametric form of a point on m	
	$\begin{pmatrix}-1+\frac{4}{7}\mu\\-2-\mu\\2-\frac{13}{7}\mu\end{pmatrix}.\begin{pmatrix}-\frac{4}{7}\\1\\\frac{13}{7}\end{pmatrix}=0$	M1: Forms scalar product of vector NP and direction vector of l and equates to zero A1: Correct vectors	M1A1
	$\Rightarrow\mu=\dfrac{56}{117}$	Solves	M1
	$\Rightarrow d=\sqrt{\left(-\frac{85}{117}\right)^2+\left(-\frac{290}{117}\right)^2+\left(\frac{10}{9}\right)^2}=\dfrac{5\sqrt{481}}{39}$	Obtains the correct distance	A1
			(5)
	Alternative for M1A1M1		
	$\overrightarrow{NP}=\begin{pmatrix}-1+\frac{4}{7}\mu\\-2-\mu\\2-\frac{13}{7}\mu\end{pmatrix}\Rightarrow d=\sqrt{\left(-1+\frac{4}{7}\mu\right)^2+(-2-\mu)^2+\left(2-\frac{13}{7}\mu\right)^2}\Rightarrow d$ is min when $\Rightarrow\mu=\dfrac{56}{117}$ M1: Find d in terms of a parameter A1: correct expression M1: use calculus (or simplify and complete the square) to find the parameter corresponding to the min d		

ALT 2	Correct vector PQ		B1
	$\begin{pmatrix}1\\2\\-2\end{pmatrix}.\begin{pmatrix}-4\\7\\13\end{pmatrix}=\begin{vmatrix}1\\2\\-2\end{vmatrix}\begin{vmatrix}-4\\7\\13\end{vmatrix}\cos\theta$	Forms the scalar product and attempts to evaluate the LHS	M1
	$\cos\theta=\dfrac{-16}{3\sqrt{234}}$	Correct value for $\cos\theta$ exact or decimal	A1
	$d=\|PQ\|\sin\theta=3\sqrt{1-\left(\dfrac{-16}{3\sqrt{234}}\right)^2}=\dfrac{5\sqrt{74}}{\sqrt{234}}$	M1: Correct method for the distance. A1: Correct **EXACT** distance	M1A1
			(5)
			Total 10

Question Number	Scheme	Notes	Marks
1(a)	$\pm\overrightarrow{AB}=\pm\begin{pmatrix}4\\-4\\-1\end{pmatrix}, \pm\overrightarrow{BC}=\pm\begin{pmatrix}-1\\5\\2\end{pmatrix}, \pm\overrightarrow{AC}=\pm\begin{pmatrix}3\\1\\1\end{pmatrix}$ Attempts any 2 of these vectors. Allow these to be written as coordinates.		M1
	E.g. $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}4\\-4\\-1\end{pmatrix}\times\begin{pmatrix}3\\1\\1\end{pmatrix}=\begin{pmatrix}-3\\-7\\16\end{pmatrix}$	Attempts the vector product of 2 appropriate vectors. If no working is shown, look for at least 2 correct elements.	**d**M1
	$\text{Area}=\frac{1}{2}\sqrt{3^2+7^2+16^2}=\frac{1}{2}\sqrt{314}$	Correct exact area. Allow recovery from sign errors in the vector product e.g. allow following a vector product of $\pm3\mathbf{i}\pm7\mathbf{j}\pm16\mathbf{k}$	A1
	Note that a correct exact area of $\frac{1}{2}\sqrt{314}$ with no evidence of any incorrect work scores full marks		
			(3)
	Alternative 1 using cosine rule:		
	$\pm\overrightarrow{AB}=\pm\begin{pmatrix}4\\-4\\-1\end{pmatrix}, \pm\overrightarrow{BC}=\pm\begin{pmatrix}-1\\5\\2\end{pmatrix}, \pm\overrightarrow{AC}=\pm\begin{pmatrix}3\\1\\1\end{pmatrix}$ Attempts any 2 of these vectors		M1
	$\lvert\pm\overrightarrow{AB}\rvert=\sqrt{4^2+4^2+1^2}, \lvert\pm\overrightarrow{BC}\rvert=\sqrt{1^2+5^2+2^2}, \lvert\pm\overrightarrow{AC}\rvert=\sqrt{3^2+1^2+1^2}$ $\cos A=\frac{33+11-30}{2\sqrt{33}\sqrt{11}}=\frac{7\sqrt{3}}{33}$ or $\cos B=\frac{30+33-11}{2\sqrt{30}\sqrt{33}}=\frac{13\sqrt{2}}{3\sqrt{55}}$ or $\cos C=\frac{30+11-33}{2\sqrt{30}\sqrt{11}}=\frac{\sqrt{8}}{\sqrt{165}}$ (For reference $A=68.44\ldots°$, $B=34.27\ldots°$, $C=77.27\ldots°$) Attempts the magnitude of all 3 sides and attempts the cosine of one of the angles using a correctly applied cosine rule **or e.g.** $\cos A=\frac{\overrightarrow{AB}.\overrightarrow{AC}}{\sqrt{33}\sqrt{11}}=\frac{12-4-1}{\sqrt{33}\sqrt{11}}$ Finds the magnitude of 2 sides and the cosine of the included angle using a correctly applied scalar product		**d**M1
	$\text{Area}=\frac{1}{2}\sqrt{11}\sqrt{33}\sin A=\frac{1}{2}\sqrt{314}$ or $\text{Area}=\frac{1}{2}\sqrt{30}\sqrt{33}\sin B=\frac{1}{2}\sqrt{314}$ or $\text{Area}=\frac{1}{2}\sqrt{30}\sqrt{11}\sin C=\frac{1}{2}\sqrt{314}$	Correct exact area. Allow recovery from sign errors in the vectors that do not affect the calculations e.g. allow $\pm\overrightarrow{AB}=\pm4\mathbf{i}\pm4\mathbf{j}\pm\mathbf{k}$, $\pm\overrightarrow{BC}=\pm\mathbf{i}\pm5\mathbf{j}\pm2\mathbf{k}$, $\pm\overrightarrow{AC}=\pm3\mathbf{i}\pm\mathbf{j}\pm\mathbf{k}$ And allow work in decimals as long as a correct exact area is found.	A1
			(3)

Alternative 2 using scalar product:		
$\pm\overrightarrow{AB} = \pm\begin{pmatrix}4\\-4\\-1\end{pmatrix}, \pm\overrightarrow{BC} = \pm\begin{pmatrix}-1\\5\\2\end{pmatrix}, \pm\overrightarrow{AC} = \pm\begin{pmatrix}3\\1\\1\end{pmatrix}$ Attempts any 2 of these vectors		M1
A to BC is $\sqrt{AB^2 - \left(\frac{\overrightarrow{AB}\bullet\overrightarrow{BC}}{BC}\right)^2} = \sqrt{\frac{157}{15}}$ **or** B to CA is $\sqrt{BC^2 - \left(\frac{\overrightarrow{BC}\bullet\overrightarrow{CA}}{CA}\right)^2} = \sqrt{\frac{314}{11}}$ **or** C to BA is $\sqrt{AC^2 - \left(\frac{\overrightarrow{AC}\bullet\overrightarrow{AB}}{AB}\right)^2} = \sqrt{\frac{314}{33}}$ Attempts one of the altitudes of triangle ABC using a correct method		**d**M1
$\text{Area} = \frac{1}{2}\sqrt{30}\sqrt{\frac{157}{15}} = \frac{1}{2}\sqrt{314}$ or $\text{Area} = \frac{1}{2}\sqrt{11}\sqrt{\frac{314}{11}} = \frac{1}{2}\sqrt{314}$ or $\text{Area} = \frac{1}{2}\sqrt{33}\sqrt{\frac{314}{33}} = \frac{1}{2}\sqrt{314}$	Correct exact area. Allow work in decimals as long as a correct exact area is found.	A1
		(3)

Alternative 3 using vector products:		
$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}0\\4\\-16\end{pmatrix}, \mathbf{b}\times\mathbf{c} = \begin{pmatrix}0\\-8\\20\end{pmatrix}, \mathbf{c}\times\mathbf{a} = \begin{pmatrix}-3\\-3\\12\end{pmatrix}$ Attempts these vector products		M1
$\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a} = \begin{pmatrix}-3\\-7\\16\end{pmatrix}$ Adds the appropriate vector products		**d**M1
$\text{Area} = \frac{1}{2}\sqrt{3^2+7^2+16^2} = \frac{1}{2}\sqrt{314}$	Correct exact area. Allow work in decimals as long as a correct exact area is found.	A1
		(3)

Question Number	Scheme	Notes	Marks
(b)	$\pm\overrightarrow{AD}=\pm\begin{pmatrix}2\\-2\\k-1\end{pmatrix}, \pm\overrightarrow{BD}=\pm\begin{pmatrix}-2\\2\\k\end{pmatrix}, \pm\overrightarrow{CD}=\pm\begin{pmatrix}-1\\-3\\k-2\end{pmatrix}$ Attempts one of these vectors		M1
	E.g. $\overrightarrow{AB}\times\overrightarrow{AC}.\overrightarrow{AD}=\begin{pmatrix}-3\\-7\\16\end{pmatrix}\bullet\begin{pmatrix}2\\-2\\k-1\end{pmatrix}=-6+14+16k-16$ E.g. $\overrightarrow{AB}\times\overrightarrow{AC}.\overrightarrow{BD}=\begin{pmatrix}-3\\-7\\16\end{pmatrix}\bullet\begin{pmatrix}-2\\2\\k\end{pmatrix}=6-14+16k$ E.g. $\overrightarrow{AB}\times\overrightarrow{AC}.\overrightarrow{CD}=\begin{pmatrix}-3\\-7\\16\end{pmatrix}\bullet\begin{pmatrix}-1\\-3\\k-2\end{pmatrix}=3+21+16k-32$ Attempts a suitable triple product to obtain a scalar quantity ($\frac{1}{6}$ not required here). They must be forming the triple product correctly e.g. not the magnitude of a vector. Do not be too concerned if they make slips as long as appropriate vectors are being used and a scalar quantity is obtained. **Must be an attempt at the tetrahedron *ABCD*.**		**d**M1
	Volume $=\frac{1}{3}\lvert 8k-4\rvert$	Correct volume. Must see modulus and must be 2 terms but allow equivalents e.g. $\frac{4}{3}\lvert 2k-1\rvert, \frac{1}{6}\lvert 16k-8\rvert, \frac{1}{6}\lvert 8-16k\rvert$ Award once a correct answer is seen and apply isw if necessary.	A1
			(3)
			Total 6

Question Number	Scheme	Notes	Marks
2(a)	$y=\ln(\tanh 2x)\Rightarrow \frac{dy}{dx}=\frac{1}{\tanh 2x}\times 2\,\text{sech}^2 2x$ **or** $y=\ln(\tanh 2x)\Rightarrow e^y=\tanh 2x\Rightarrow e^y\frac{dy}{dx}=2\,\text{sech}^2 2x\Rightarrow\frac{dy}{dx}=\frac{2\,\text{sech}^2 2x}{\tanh 2x}$ M1: Applies the chain rule or eliminates the "ln" and differentiates implicitly to obtain to obtain $\frac{dy}{dx}=\frac{k\,\text{sech}^2 2x}{\tanh 2x}$ A1: Correct derivative in any form **Note that some candidates now convert to exponential form to complete this part – see below in the alternative for scoring the final M1A1**		M1A1
	$=\frac{2\cosh 2x}{\sinh 2x}\times\frac{1}{\cosh^2 2x}=\frac{2}{\sinh 2x\cosh 2x}$	Converts to sinh2x and cosh2x correctly to obtain $\frac{k}{\sinh 2x\cosh 2x}$	M1
	$=\frac{2}{\frac{1}{2}\sinh 4x}=4\text{cosech}4x$	Correct answer. Note that this is not a given answer so you can allow if e.g. a sinh becomes a sin but is then recovered but if there are any obvious errors this mark should be withheld.	A1
			(4)
Alternative using exponentials:			
	$y=\ln(\tanh 2x)=\ln\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)$ $\frac{dy}{dx}=\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}\left(\frac{(e^{2x}+e^{-2x})(2e^{2x}+2e^{-2x})-(e^{2x}-e^{-2x})(2e^{2x}-2e^{-2x})}{(e^{2x}+e^{-2x})^2}\right)$ **or** $y=\ln(\tanh 2x)=\ln\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)=\ln(e^{2x}-e^{-2x})-\ln(e^{2x}+e^{-2x})$ $\frac{dy}{dx}=\frac{2e^{2x}+2e^{-2x}}{e^{2x}-e^{-2x}}-\frac{2e^{2x}-2e^{-2x}}{e^{2x}+e^{-2x}}$ M1: Writes tanh2x correctly in terms of exponentials and applies the chain rule and quotient rule or uses the subtraction law of logs and applies the chain rule A1: Correct derivative in any form		M1A1
	$=\frac{2(e^{2x}+e^{-2x})^2-2(e^{2x}-e^{-2x})^2}{e^{4x}-e^{-4x}}=\frac{8}{e^{4x}-e^{-4x}}$ Obtains $\frac{k}{e^{4x}-e^{-4x}}$		M1
	$=\frac{4}{\sinh 4x}=4\text{cosech}4x$	Correct answer. Note that this is not a given answer so you can allow if e.g. a sinh becomes a sin but is then recovered but if there are any obvious errors this mark should be withheld.	A1

(b) Way 1	$4\text{cosech}4x = 1 \Rightarrow \sinh 4x = 4 \Rightarrow 4x = \ln\left(4+\sqrt{4^2+1}\right)$ Changes to sinh 4x =… and uses the **correct** logarithmic form of arsinh to reach 4x = …		M1
	$x = \frac{1}{4}\ln\left(4+\sqrt{17}\right)$	This value only. Allow e.g. $x = \ln\left(4+\sqrt{17}\right)^{\frac{1}{4}}$	A1
			(2)
(b) Way 2	$4\text{cosech}4x = 1 \Rightarrow 4\times\frac{2}{e^{4x}-e^{-4x}} = 1 \Rightarrow e^{8x} - 8e^{4x} - 1 = 0$ Changes to the **correct** exponential form to reach $\frac{k}{e^{4x}-e^{-4x}}$, obtains a 3TQ in e^{4x}, solves and takes ln's to reach 4x = … (usual rules for solving a 3TQ do not apply as long as the above conditions are met)		M1
	$x = \frac{1}{4}\ln\left(4+\sqrt{17}\right)$	This value only. Allow e.g. $x = \ln\left(4+\sqrt{17}\right)^{\frac{1}{4}}$	A1
			(2)
			Total 6

Question Number	Scheme	Notes	Marks
3(a)	$\mathbf{A}=\begin{pmatrix}2 & k & 2\\ 2 & 2 & k\\ 1 & 2 & 2\end{pmatrix}$		
	$\lvert\mathbf{A}\rvert=2(4-2k)-k(4-k)+2(4-2)=0$ $\Rightarrow k^2-8k+12=0\Rightarrow k=\ldots$ Attempts det $\mathbf{A}=0$ and solves 3TQ to obtain 2 values for k Note that the usual rules for solving a 3TQ do not need to be applied as long as 2 values for k are obtained. The attempt at the determinant should be a correct expression for their row or column so allow errors only when collecting terms Note that the rule of Sarrus gives $8+k^2+8-4-4k-4k=0$		M1
	$k=2,\ 6$	Correct values.	A1
	Marks for part (a) can only be scored in their attempt at (a) and not recovered from part (b)		
			(2)
(b)	$\begin{pmatrix}2 & k & 2\\ 2 & 2 & k\\ 1 & 2 & 2\end{pmatrix}\rightarrow\begin{pmatrix}4-2k & 4-k & 2\\ 2k-4 & 2 & 4-k\\ k^2-4 & 2k-4 & 4-2k\end{pmatrix}\rightarrow\begin{pmatrix}4-2k & k-4 & 2\\ 4-2k & 2 & k-4\\ k^2-4 & 4-2k & 4-2k\end{pmatrix}$ Applies the correct method to reach at least a matrix of cofactors Should be an attempt at the minors followed by $\begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix}$ If there is any doubt then look for at least 6 correct cofactors		M1
	$\begin{pmatrix}4-2k & k-4 & 2\\ 4-2k & 2 & k-4\\ k^2-4 & 4-2k & 4-2k\end{pmatrix}\rightarrow\begin{pmatrix}4-2k & 4-2k & k^2-4\\ k-4 & 2 & 4-2k\\ 2 & k-4 & 4-2k\end{pmatrix}$ **d**M1: Attempts adjoint matrix by transposing. Dependent on previous mark. A1: Correct adjoint		**d**M1 A1
	$\mathbf{A}^{-1}=\dfrac{1}{k^2-8k+12}\begin{pmatrix}4-2k & 4-2k & k^2-4\\ k-4 & 2 & 4-2k\\ 2 & k-4 & 4-2k\end{pmatrix}$ Fully correct inverse **or** follow through their incorrect determinant **from part (a)** where their determinant is a function of k		A1ft
	Ignore any labelling of the matrices and allow any type of brackets around the matrices		
			(4)
			Total 6

Question Number	Scheme	Notes	Marks
4	$x=4\cosh\theta \Rightarrow \frac{dx}{d\theta}=4\sinh\theta$ $\Rightarrow \int \frac{1}{(x^2-16)^{\frac{3}{2}}}dx=\int \frac{4\sinh\theta}{(16\cosh^2\theta-16)^{\frac{3}{2}}}d\theta$ Full attempt to use the given substitution. Award for $\int \frac{1}{(x^2-16)^{\frac{3}{2}}}dx=k\int \frac{\sinh\theta}{((4\cosh\theta)^2-16)^{\frac{3}{2}}}d\theta$ Condone $4\cosh^2\theta$ for $(4\cosh\theta)^2$		M1
	$=\int \frac{4\sinh\theta}{(16\sinh^2\theta)^{\frac{3}{2}}}d\theta=\int \frac{4\sinh\theta}{64\sinh^3\theta}d\theta$ Simplifies $(16\cosh^2\theta-16)^{\frac{3}{2}}$ to the form $k\sinh^3\theta$ which may be implied by: $\int \frac{1}{(x^2-16)^{\frac{3}{2}}}dx=k\int \frac{1}{\sinh^2\theta}d\theta$ **Note that this is not dependent on the first M**		M1
	$=\int \frac{1}{16\sinh^2\theta}d\theta$ Fully correct simplified integral. Allow equivalents e.g. $\frac{1}{16}\int \operatorname{cosech}^2\theta\, d\theta$, $\int \frac{1}{(4\sinh\theta)^2}d\theta$, $\int (4\sinh\theta)^{-2}d\theta$ etc. May be implied by subsequent work.		A1
	$=\int \frac{1}{16\sinh^2\theta}d\theta=\frac{1}{16}\int \operatorname{cosech}^2\theta\, d\theta=-\frac{1}{16}\coth\theta(+c)$ Integrates to obtain $k\coth\theta$. **Depends on both previous method marks.**		**d**M1
	$=-\frac{1}{16}\frac{\cosh\theta}{\sinh\theta}+c=-\frac{1}{16}\frac{\frac{x}{4}}{\sqrt{\frac{x^2}{16}-1}}+c$ or e.g. $-\frac{1}{4}\frac{\frac{x}{4}}{\sqrt{x^2-16}}+c$ Substitutes back **correctly** for x by replacing $\cosh\theta$ with $\frac{x}{4}$ or equivalent e.g. $4\cosh\theta$ with x **and** $\sinh\theta$ with $\sqrt{\left(\frac{x}{4}\right)^2-1}$ or equivalent e.g. $4\sinh\theta$ with $\sqrt{x^2-16}$ **Depends on all previous method marks and must be fully correct work for their** "$-\frac{1}{16}$"		**d**M1
	$\frac{-x}{16\sqrt{x^2-16}}(+c)$ oe e.g. $\frac{-\frac{1}{16}x}{\sqrt{x^2-16}}(+c)$	Correct answer. Award once the correct answer is seen and apply isw if necessary. Condone the omission of "+ c"	A1
	Note that you can condone the omission of the "$d\theta$" throughout		
			(6)
			Total 6

Question Number	Scheme	Notes	Marks
	Mark (a) and (b) together but do not credit work for (a) that is seen in (c)		
5(a)	$\begin{pmatrix}6&-2&-1\\-2&6&-1\\-1&-1&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}8x\\8y\\8z\end{pmatrix}$ or $\begin{pmatrix}-2&-2&-1\\-2&-2&-1\\-1&-1&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\ldots$ Correct method for obtaining the eigenvector		M1
	$\mathbf{i}-\mathbf{j}$	Any multiple of this vector	A1
			(2)
(b)	$\lvert\mathbf{M}-\lambda\mathbf{I}\rvert=\begin{vmatrix}6-\lambda&-2&-1\\-2&6-\lambda&-1\\-1&-1&5-\lambda\end{vmatrix}$ $\Rightarrow\underline{(6-\lambda)}\underline{\underline{((6-\lambda)(5-\lambda)-1)}}\underline{+2}\underline{\underline{(2(\lambda-5)-1)}}\ \underline{-1}\underline{\underline{(2+6-\lambda)}}$ Correct attempt at the determinant of $\mathbf{M}-\lambda\mathbf{I}$. The terms with single underlining should be correct with correct signs but allow minor slips in the brackets with double underlining. Note that the rule of Sarrus gives $(6-\lambda)(6-\lambda)(5-\lambda)-2-2-(6-\lambda)-(6-\lambda)-4(5-\lambda)$		M1
	$\Rightarrow\lambda^3-17\lambda^2+90\lambda-144=0\Rightarrow\lambda=\ldots$	Solves $\mathbf{M}-\lambda\mathbf{I}=0$ to obtain 2 different distinct real eigenvalues excluding 8	M1
	$\Rightarrow\lambda=3,6,(8)$	For 3 and 6	A1
			(3)

Question Number	Scheme	Notes	Marks
(c)	$(\mathbf{D}=)\begin{pmatrix}8&0&0\\0&3&0\\0&0&6\end{pmatrix}$	Correct **D** with distinct non-zero eigenvalues in any order. Follow through their non-zero 3 and 6. Ignore labelling and score for sight of the correct or correct ft matrix.	B1ft
	$\begin{pmatrix}6&-2&-1\\-2&6&-1\\-1&-1&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3x\\3y\\3z\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\ldots$ NB $\mathbf{v}_2=k\begin{pmatrix}1\\1\\1\end{pmatrix}$ **and** $\begin{pmatrix}6&-2&-1\\-2&6&-1\\-1&-1&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6x\\6y\\6z\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\ldots$ NB $\mathbf{v}_3=k\begin{pmatrix}1\\1\\-2\end{pmatrix}$ Attempts eigenvectors for their other 2 distinct eigenvalues not including 8 May use e.g. $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$		M1
	$(\mathbf{P}=)\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&\frac{1}{\sqrt{3}}&-\frac{2}{\sqrt{6}}\end{pmatrix}$ Forms a complete **P** from normalised eigenvectors using their eigenvector from part (a) and their other 2 eigenvectors formed from their other 2 different distinct eigenvalues in any order. Ignore labelling and score for forming this matrix which may be seen as part of a calculation.		M1
	$\mathbf{D}=\begin{pmatrix}8&0&0\\0&3&0\\0&0&6\end{pmatrix}$ **and** $\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&\frac{1}{\sqrt{3}}&-\frac{2}{\sqrt{6}}\end{pmatrix}$ All fully correct and consistent and correctly labelled but the labelling may be implied by their working.		A1
			(4)
			Total 9

Question Number	Scheme	Notes	Marks
6(a) Way 1	$\int \frac{x^n}{\sqrt{x^2+3}}\,dx = \int x^{n-1}x\left(x^2+3\right)^{-\frac{1}{2}}\,dx$ **or** $\int \frac{x^n}{\sqrt{x^2+3}}\,dx = \int x^{n-1}\,d\left(x^2+3\right)^{\frac{1}{2}}$ **Applies** $x^n = x^{n-1}\times x$ to $\int \frac{x^n}{\sqrt{x^2+3}}\,dx$ but may be implied by subsequent work		M1
	$\int x^{n-1}x\left(x^2+3\right)^{-\frac{1}{2}}\,dx = x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \int (n-1)x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\,dx$ **d**M1: Applies integration by parts to obtain $\alpha x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \beta\int x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\,dx$ (NB α, β may be functions of n) Note that if a correct formula for parts is quoted first and parts is applied in the correct direction then we can condone slips in signs as long as the expression is of the above form. **If you are unsure – send to review.** A1: Correct expression		**d**M1A1
	$= x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \int (n-1)x^{n-2}\left(x^2+3\right)\left(x^2+3\right)^{-\frac{1}{2}}\,dx$ Applies $\left(x^2+3\right)^{\frac{1}{2}} = \left(x^2+3\right)\left(x^2+3\right)^{-\frac{1}{2}}$ **having made an attempt at integration by parts in the correct direction**		M1
	$= x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - (n-1)\int x^n\left(x^2+3\right)^{-\frac{1}{2}}\,dx - 3(n-1)\int x^{n-2}\left(x^2+3\right)^{-\frac{1}{2}}\,dx$ $= x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - (n-1)I_n - 3(n-1)I_{n-2}$ Splits into 2 integrals involving I_n and I_{n-2} **Depends on all the previous method marks**		**d**M1
	$\Rightarrow I_n = \frac{x^{n-1}}{n}\left(x^2+3\right)^{\frac{1}{2}} - \frac{3(n-1)}{n}I_{n-2}$ * Obtains the printed answer. You can condone the odd missing "dx" but if there are any clear errors e.g. invisible brackets that are not recovered, sign errors etc. then this mark should be withheld.		A1*
			(6)

Question Number	Scheme	Notes	Marks
6(a) Way 2	$\int \frac{x^n}{\sqrt{x^2+3}}\mathrm{d}x = \int x^{n-2}x^2\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x$ Applies $x^n = x^{n-2} \times x^2$		M1
	$\int x^{n-2}x^2\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x = \int x^{n-2}\left(x^2+3-3\right)\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x$ $= \int x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\mathrm{d}x - \int 3x^{n-2}\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x$ **d**M1: Writes x^2 as $\left(x^2+3-3\right)$ to obtain $\alpha\int x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\mathrm{d}x - \beta\int x^{n-2}\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x$ A1: Correct expression		**d**M1A1
	$\int x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\mathrm{d}x = \frac{x^{n-1}}{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \frac{1}{n-1}\int x^n\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x$ Applies integration by parts on $\int x^{n-2}\left(x^2+3\right)^{\frac{1}{2}}\mathrm{d}x$ to obtain $\alpha x^{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \beta\int x^n\left(x^2+3\right)^{-\frac{1}{2}}\mathrm{d}x$ Note that if a correct formula for parts is quoted first and parts is applied in the correct direction then we can condone slips in signs as long as the expression is of the above form. **If you are unsure – send to review.**		M1
	$I_n = \frac{x^{n-1}}{n-1}\left(x^2+3\right)^{\frac{1}{2}} - \frac{1}{n-1}I_n - 3I_{n-2}$ Brings all together and introduces I_n and I_{n-2} Depends on all the previous method marks		**d**M1
	$\Rightarrow I_n = \frac{x^{n-1}}{n}\left(x^2+3\right)^{\frac{1}{2}} - \frac{3(n-1)}{n}I_{n-2}$ * Obtains the printed answer. You can condone the odd missing "dx" but if there are any clear errors e.g. invisible brackets that are not recovered, sign errors etc. then this mark should be withheld.		A1*

Question Number	Scheme	Notes	Marks
(b) Way 1	$I_5 = \frac{x^4}{5}(x^2+3)^{\frac{1}{2}} - \frac{12}{5}I_3$ Applies the reduction formula once to obtain I_5 in terms of I_3 Allow slips on coefficients only		M1
	$I_5 = \frac{x^4}{5}(x^2+3)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}(x^2+3)^{\frac{1}{2}} - \frac{6}{3}I_1\right)$ Applies the reduction formula again to obtain an expression for I_5 in terms of I_1 and allow "I_1"or what they think is I_1 Allow slips on coefficients only		M1
	E.g. $I_5 = \frac{x^4}{5}(x^2+3)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}(x^2+3)^{\frac{1}{2}} - \frac{6}{3}(x^2+3)^{\frac{1}{2}}\right)$ Or e.g. $I_5 = \frac{x^4}{5}(x^2+3)^{\frac{1}{2}} - \frac{4}{5}x^2(x^2+3)^{\frac{1}{2}} + \frac{24}{5}(x^2+3)^{\frac{1}{2}}$ **Any** correct expression in terms of x only		A1
	$I_5 = \frac{1}{5}(x^2+3)^{\frac{1}{2}}(x^4-4x^2+24)+k$ Must include the "+ k" but allow other letter e.g. + c		A1
			(4)
			Total 10

Question Number	Scheme	Marks
(b) Way 2	NB $I_1 = (x^2+3)^{\frac{1}{2}}$	
	$I_3 = \frac{x^2}{3}(x^2+3)^{\frac{1}{2}} - \frac{6}{3}I_1$ Applies the reduction formula once to obtain I_3 in terms of I_1 and allow "I_1" or what they think is I_1 Allow slips on coefficients only	M1
	$I_5 = \frac{x^4}{5}(x^2+3)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}(x^2+3)^{\frac{1}{2}} - 2I_1\right)$ Applies the reduction formula again to obtain an expression for I_5 in terms of I_1 and allow "I_1" or what they think is I_1 Allow slips on coefficients only	M1
	E.g. $I_5 = \frac{x^4}{5}(x^2+3)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{x^2}{3}(x^2+3)^{\frac{1}{2}} - \frac{6}{3}(x^2+3)^{\frac{1}{2}}\right)$ Or e.g. $I_5 = \frac{x^4}{5}(x^2+3)^{\frac{1}{2}} - \frac{4}{5}x^2(x^2+3)^{\frac{1}{2}} + \frac{24}{5}(x^2+3)^{\frac{1}{2}}$ **Any** correct expression in terms of x only	A1
	$I_5 = \frac{1}{5}(x^2+3)^{\frac{1}{2}}(x^4-4x^2+24)+k$ Must include the "+ k" but allow other letter e.g. + c	A1

Question Number	Scheme	Notes	Marks
(b) Way 3	$I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{12}{5}I_3$ Applies the reduction formula once to obtain I_5 in terms of I_3 Allow slips on coefficients only		M1
	$I_3 = \int \frac{x^3}{\left(x^2+3\right)^{\frac{1}{2}}}\,dx$ $u = x^2+3 \Rightarrow I_3 = \int \frac{(u-3)^{\frac{3}{2}}}{u^{\frac{1}{2}}} \frac{du}{2(u-3)^{\frac{1}{2}}} = \frac{1}{2}\int \frac{(u-3)}{u^{\frac{1}{2}}}\,du = \frac{1}{3}u^{\frac{3}{2}} - 6u^{\frac{1}{2}}$ $= \frac{1}{3}\left(x^2+3\right)^{\frac{3}{2}} - 6\left(x^2+3\right)^{\frac{1}{2}}$ $I_5 = \frac{x^4}{5}\left(x^2+3\right)^{\frac{1}{2}} - \frac{12}{5}\left(\frac{1}{3}\left(x^2+3\right)^{\frac{3}{2}} - 6\left(x^2+3\right)^{\frac{1}{2}}\right)$ M1: A credible attempt to find I_3 and then expresses I_5 in terms of x A1: **Any** correct expression in terms of x only		M1A1
	$I_5 = \frac{1}{5}\left(x^2+3\right)^{\frac{1}{2}}\left(x^4-4x^2+24\right)+k$ Must include the "+ k" but allow other letter e.g. + c		A1

Question Number	Scheme	Notes	Marks
7(a)	$5\mathbf{i}+3\mathbf{j}-8\mathbf{k}$ and $2\mathbf{i}-3\mathbf{j}-6\mathbf{k}$ lie in Π_1	Identifies 2 correct vectors lying in Π_1	B1
	$\mathbf{n}=\begin{pmatrix}5\\3\\-8\end{pmatrix}\times\begin{pmatrix}2\\-3\\-6\end{pmatrix}=\begin{pmatrix}-18-24\\-(-30+16)\\-15-6\end{pmatrix}$ Attempts the vector product between 2 **correct** vectors in Π_1 If no working is shown, look for at least 2 correct elements. Or e.g. Let $\mathbf{n}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ then $(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\bullet(5\mathbf{i}+3\mathbf{j}-8\mathbf{k})=0,\ (a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\bullet(2\mathbf{i}-3\mathbf{j}-6\mathbf{k})=0$ $\Rightarrow 5a+3b-8c=0,\ 2a-3b-6c=0\Rightarrow a=2c,\ 3b=-2c\Rightarrow\mathbf{n}=...$		M1
	$=\begin{pmatrix}-42\\14\\-21\end{pmatrix}$ or e.g. $\begin{pmatrix}6\\-2\\3\end{pmatrix}$	Correct normal vector	A1
	$(6\mathbf{i}-2\mathbf{j}+3\mathbf{k})\bullet(\mathbf{i}+2\mathbf{j}+\mathbf{k})=...$ Attempts scalar product between their normal vector and position vector of a point in Π_1. Do not allow this mark if the "5" (or equivalent) just 'appears'. There must be some evidence for its origin e.g. **a**.**n** = … where **a** and **n** have been defined earlier. **Depends on the first method mark.**		**d**M1
	$6x-2y+3z=5$*	Correct proof	A1*
			(5)
	Alternative 1 for (a):		
	E.g. Let equation of Π_1 be $ax+by+z=c$ 3 points on Π_1 are (1, 2, 1), (3, −1, −5) and e.g. (8, 2, −13)		B1
	$a+2b+1=c,\ 3a-b-5=c,\ 8a+2b-13=c\Rightarrow a=...,b=...,c=...$ Solves simultaneously for a, b and c using **correct** points		M1
	$\Rightarrow a=2, b=-\frac{2}{3}, c=\frac{5}{3}$	Correct values	A1
	$2x-\frac{2}{3}y+z=\frac{5}{3}$	Forms Cartesian equation	**d**M1
	$6x-2y+3z=5$*	Correct proof	A1*
	Alternative 2 for (a):		
	$(1,2,1)\rightarrow 6x-2y+3z=6-4+3=5$ Shows (1, 2, 1) lies on Π_1		B1
	$\frac{x-3}{5}=\frac{y+1}{3}=\frac{z+5}{-8}\rightarrow\mathbf{r}=\begin{pmatrix}3\\-1\\-5\end{pmatrix}+\lambda\begin{pmatrix}5\\3\\-8\end{pmatrix}$ or equivalent M1: Converts l to **correct** parametric form <u>**seen as part of an attempt at this alternative**</u> allow 1 slip with one of the elements A1: Correct form		M1A1
	$6(3+5\lambda)-2(-1+3\lambda)+3(-5-8\lambda)=5$		**d**M1

	Shows l lies in Π_1		
	P lies in Π_1 and l lies in Π_1 so $6x-2y+3z=5$* All correct with conclusion		A1*
(b) Way 1	$d=\frac{\lvert 6(2)-2k+3(-7)-5\rvert}{\sqrt{6^2+2^2+3^2}}$	Correct method for the shortest distance	M1
	$=\frac{1}{7}\lvert -2k-14\rvert=\frac{2}{7}\lvert k+7\rvert$*	Correct completion	A1*
			(2)
(b) Way 2	Distance O to Π_1 is $\frac{5}{\sqrt{6^2+2^2+3^2}}$. Distance O to parallel plane containing Q is $\frac{(6\mathbf{i}-2\mathbf{j}+3\mathbf{k})\bullet(2\mathbf{i}+k\mathbf{j}-7\mathbf{k})}{\sqrt{6^2+2^2+3^2}}=\frac{-9-2k}{7}$ $d=\left\lvert\frac{5}{7}-\frac{-9-2k}{7}\right\rvert$ Correct method for the shortest distance		M1
	$=\frac{1}{7}\lvert 2k+14\rvert=\frac{2}{7}\lvert k+7\rvert$*	Correct completion	A1*
(b) Way 3	$d=\left\lvert\frac{\overrightarrow{PQ}\bullet\mathbf{n}}{\lvert\mathbf{n}\rvert}\right\rvert=\left\lvert\frac{(\mathbf{i}+(k-2)\mathbf{j}-8\mathbf{k})\bullet(-42\mathbf{i}+14\mathbf{j}-21\mathbf{k})}{\sqrt{42^2+14^2+21^2}}\right\rvert$ Correct method for the shortest distance		M1
	$=\left\lvert\frac{-42+14k-28+168}{49}\right\rvert=\left\lvert\frac{14k+98}{49}\right\rvert=\frac{2}{7}\lvert k+7\rvert$*	Correct completion	A1*
(c)	$\frac{2}{7}\lvert k+7\rvert=\frac{\lvert 8(2)-4k-7+3\rvert}{\sqrt{8^2+4^2+1^2}}$ Correctly attempts the distance between (2, k, −7) and Π_2 and sets equal to the result from (a). May see alternative methods here for the distance between (2, k, −7) and Π_2 e.g. finds the coordinates of a point on Π_2 e.g. R(1, 1, −7) and then finds $d=\left\lvert\frac{\overrightarrow{RQ}\bullet(8\mathbf{i}-4\mathbf{j}+\mathbf{k})}{\lvert 8\mathbf{i}-4\mathbf{j}+\mathbf{k}\rvert}\right\rvert=\left\lvert\frac{(\mathbf{i}+(k-1)\mathbf{j})\bullet(8\mathbf{i}-4\mathbf{j}+\mathbf{k})}{\sqrt{8^2+4^2+1^2}}\right\rvert=\left\lvert\frac{8-4k+4}{9}\right\rvert=\left\lvert\frac{12-4k}{9}\right\rvert$		M1
	$\frac{2}{7}(k+7)=\text{"}\frac{1}{9}(12-4k)\text{"}\Rightarrow k=\ldots$ or $\frac{2}{7}(k+7)=\text{"}\frac{1}{9}(4k-12)\text{"}\Rightarrow k=\ldots$ Attempts to solve one of these equations where their distance from Q to Π_2 is of the form ak + b where a and b are non-zero. **or** $\frac{2}{7}(k+7)=\text{"}\frac{1}{9}(12-4k)\text{"}\Rightarrow\frac{4}{49}(k+7)^2=\text{"}\frac{1}{81}(12-4k)^2\text{"}$ $\Rightarrow 23k^2-462k-441=0\Rightarrow k=\ldots$ Squares both sides and attempts to solve resulting quadratic. Condone poor attempts at squaring the brackets and there is no requirement to follow the usual guidance for solving the quadratic		**d**M1
	$k=-\frac{21}{23}$ **or** $k=21$	One correct value. Must be 21 but allow equivalent exact fractions for $-\frac{21}{23}$	A1

	$k=-\frac{21}{23}$ **and** $k=21$	Both correct values. Must be 21 but allow equivalent exact fractions for $-\frac{21}{23}$ and no other values.	A1
			(4)
			Total 11

Question Number	Scheme	Notes	Marks
8(a)	$\frac{dy}{dx}=\frac{-2x}{1-x^2}$	Correct derivative	B1
	$1+\left(\frac{dy}{dx}\right)^2=1+\frac{4x^2}{(1-x^2)^2}=\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}$ or $\frac{x^4-2x^2+1+4x^2}{(1-x^2)^2}$ or $\frac{x^4+2x^2+1}{(1-x^2)^2}$ Attempts $1+\left(\frac{dy}{dx}\right)^2$, finds common denominator and shows working in the numerator condoning sign slips only. (The denominator may be expanded)		M1
	$=\frac{(1+x^2)^2}{(1-x^2)^2}$ or $\left(\frac{1+x^2}{1-x^2}\right)^2$	Fully correct expression with factorised numerator and denominator.	A1
	$\int_{\frac{1}{2}}^{\frac{3}{4}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\frac{1+x^2}{1-x^2}\right)dx$ *	Fully correct proof with no errors and integral as printed on the question paper but allow x^2+1 for $1+x^2$ and allow $\int_{\frac{1}{2}}^{\frac{3}{4}}\frac{(1+x^2)}{(1-x^2)}\,dx$ or $\int_{\frac{1}{2}}^{\frac{3}{4}}\frac{1+x^2}{1-x^2}\,dx$	A1*
			(4)

Question Number	Scheme	Notes	Marks
(b)	$\frac{(x^2+1)}{(1-x^2)} = -1+\frac{2}{1-x^2}$ or e.g. $-1+\frac{1}{1-x}+\frac{1}{1+x}$ Writes the improper fraction correctly		B1
	$\int \frac{k}{1-x^2}\,dx = \pm\alpha \ln\frac{1+x}{1-x}$ **Or e.g.** $\int \frac{k}{1-x^2}\,dx = \pm\alpha\ln(1+x)\pm\alpha\ln(1-x)$ Achieves an acceptable **logarithmic** form for $\int \frac{k}{1-x^2}\,dx$ (k constant) (may see partial fraction approach). If they use artanh here, this mark and the next mark will become available when they change to logarithmic form e.g. when they substitute the limits later.		M1
	$\int -1+\frac{2}{1-x^2}\,dx = -x+\ln\frac{1+x}{1-x}$	Correct integration	A1
	$\left[-x+\ln\frac{1+x}{1-x}\right]_{\frac{1}{2}}^{\frac{3}{4}} = -\frac{3}{4}+\ln 7-\left(-\frac{1}{2}+\ln 3\right)$	Evidence that the given limits have been applied. Condone slips as long as the intention is clear. **Depends on the previous M.**	**d**M1
	$= -\frac{1}{4}+\ln\frac{7}{3}$	cao	A1
			(5)
	Note that a common incorrect approach is: $\int \frac{(1+x^2)}{(1-x^2)}\,dx = \int\left(\frac{1}{1-x^2}+\frac{x^2}{1-x^2}\right)dx = \frac{1}{2}\ln\frac{1+x}{1-x}+\ldots$ $=\left[\frac{1}{2}\ln\frac{1+x}{1-x}+\ldots\right]_{\frac{1}{2}}^{\frac{3}{4}} = \ldots$ If there is no attempt at $\int\left(\frac{x^2}{1-x^2}\right)dx$ this will generally score B0M1A0M0A0 BUT If there is an attempt at $\int\left(\frac{x^2}{1-x^2}\right)dx$ (however poor) and evidence that the limits have been applied this will generally score B0M1A0M1A0. Condone slips with the substitution of limits as long as the intention is clear. **BUT** note that attempts that consider partial fractions such as $\frac{1+x^2}{1-x^2}\equiv\frac{A}{1-x}+\frac{B}{1+x}$ will generally score no marks – **if you are unsure, send to review.** Note also that $\frac{1+x^2}{1-x^2}\equiv\frac{A}{1-x}+\frac{B}{1+x}+C$ is a correct form and could score full marks. Also, use of $\frac{(1+x^2)}{(1-x^2)}=\frac{1-x^2+2x^2}{1-x^2}=1+\frac{2x^2}{1-x^2}$ with no attempt to deal with the $\frac{2x^2}{1-x^2}$ as an improper fraction as in the main scheme is likely to score no marks.		
			Total 9

Alternative approach to integration in part (b) by substitution:

(b)	$x=\tanh\theta \Rightarrow \int \frac{(1+x^2)}{(1-x^2)}\,dx = \int \frac{(1+\tanh^2\theta)}{(1-\tanh^2\theta)}\operatorname{sech}^2\theta\,d\theta$ Substitutes fully		B1
	$\int \frac{(1+\tanh^2\theta)}{(1-\tanh^2\theta)}\operatorname{sech}^2\theta\,d\theta = \int (1+\tanh^2\theta)\,d\theta$ $= \int (2-\operatorname{sech}^2\theta)\,d\theta$ Cancel and applies $\tanh^2\theta = 1-\operatorname{sech}^2\theta$		M1
	$= \int (2-\operatorname{sech}^2\theta)\,d\theta = 2\theta - \tanh\theta$	Correct integration	A1
	$[2\text{artanh}x - x]_{\frac{1}{2}}^{\frac{3}{4}} = 2\times\frac{1}{2}\ln\left(\frac{1+\frac{3}{4}}{1-\frac{3}{4}}\right) - \frac{3}{4} - \left(2\times\frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) - \frac{1}{2}\right)$ Evidence that the given limits have been applied. Condone slips as long as the intention is clear. **Depends on the previous M.**		**d**M1
	$= -\frac{1}{4} + \ln\frac{7}{3}$	cao	A1
			(5)

Note that a similar approach can be applied to $\int \left(\frac{x^2}{1-x^2}\right) dx$

Question Number	Scheme	Notes	Marks
9	$\dfrac{x^2}{25}+\dfrac{y^2}{16}=1,\quad (5\cos\theta, 4\sin\theta)$		
(a)	$\dfrac{dx}{d\theta}=-5\sin\theta,\ \dfrac{dy}{d\theta}=4\cos\theta$ **or** $\dfrac{2x}{25}+\dfrac{2y}{16}\dfrac{dy}{dx}=0$ oe **or** $\dfrac{dy}{dx}=-\dfrac{4x}{25}\left(1-\dfrac{x^2}{25}\right)^{-\frac{1}{2}}$ oe	Correct derivatives or correct implicit differentiation or correct explicit differentiation.	B1
	$\dfrac{dy}{dx}=\dfrac{4\cos\theta}{-5\sin\theta}$	Divides their derivatives correctly or substitutes and rearranges	M1
	$M_N=\dfrac{5\sin\theta}{4\cos\theta}$	Correct perpendicular gradient rule – may be implied when they form the normal equation.	M1
	$y-4\sin\theta=\dfrac{5\sin\theta}{4\cos\theta}(x-5\cos\theta)$	Correct straight line method (any complete method). **Must** use their gradient of the normal.	M1
	$5x\sin\theta-4y\cos\theta=9\sin\theta\cos\theta$* **or** $9\sin\theta\cos\theta=5x\sin\theta-4y\cos\theta$*	Achieves the printed answer with no errors and allow this answer to be obtained from the previous line. Allow $5\sin\theta x$ for $5x\sin\theta$ and $4\cos\theta y$ for $4y\cos\theta$.	A1*
	Allow all marks if the gradient is seen as a function of x and y initially (even in the straight line equation) as long as this is recovered correctly.		
	Solutions that do not use calculus e.g. just quoting the equation of the normal as $y-4\sin\theta=\dfrac{5\sin\theta}{4\cos\theta}(x-5\cos\theta)$ **send to review however if they just quote e.g.** $ax\sin\theta-by\sin\theta=(a^2-b^2)\sin\theta\cos\theta$ **and then write down the given result this scores no marks.** **But we would accept** $\dfrac{dy}{dx}=\dfrac{4\cos\theta}{-5\sin\theta}$ **to be quoted for a full solution.**		
			(5)
(b)	$b^2=a^2(1-e^2)\Rightarrow 16=25(1-e^2)\Rightarrow e=\dfrac{3}{5}$ F is $(ae,\ 0)=\left(5\times\dfrac{3}{5},\ 0\right)$ Or e.g. $"c"^2=a^2e^2=a^2-b^2=25-16\Rightarrow a^2e^2=9\Rightarrow ae=\ldots$ Fully correct strategy for F (must be numerical so $(5e, 0)$ is M0		M1
	(3, 0)	Correct coordinates. (±3, 0) scores A0	A1
			(2)

(c)	$x = \frac{9}{5}\cos\theta$	Correct x coordinate (of Q)	B1
	$PF^2 = (5\cos\theta - "3")^2 + (4\sin\theta)^2$ or $PF = \sqrt{(5\cos\theta - "3")^2 + (4\sin\theta)^2}$	Correct application of Pythagoras to find PF or PF^2. Their "3" should be positive but allow work in terms of e e.g. "5e".	M1
	$= 25\cos^2\theta - 30\cos\theta + 9 + 16\sin^2\theta$ $= 25\cos^2\theta - 30\cos\theta + 9 + 16(1-\cos^2\theta)$	Applies $\sin^2\theta = 1 - \cos^2\theta$ to obtain a quadratic expression in $\cos\theta$. If the correct identity is not seen explicitly then their working must imply that a correct identity has been used. **Depends on the previous M.**	**d**M1
	$PF = \pm(5 - 3\cos\theta)$ $PF^2 = 9\cos^2\theta - 30\cos\theta + 25$	Correct expression for PF or PF^2 in terms of $\cos\theta$ with terms collected.	A1
	Note that an alternative to using Pythagoras to find PF is to use $PF = ePM$ where M is the foot of the perpendicular from P to the **positive** directrix. Score M1 for $x = \frac{a}{e} = \frac{5}{3/5}\left(= \frac{25}{3}\right)$ (not $\pm\frac{25}{3}$) and **d**M1A1 for $PF = ePM = \frac{3}{5}\left(\frac{25}{3} - 5\cos\theta\right)$		
	$\frac{\lvert QF\rvert}{\lvert PF\rvert} = \frac{3 - \frac{9}{5}\cos\theta}{5 - 3\cos\theta} = \frac{3\left(1 - \frac{3}{5}\cos\theta\right)}{5\left(1 - \frac{3}{5}\cos\theta\right)}$ or e.g. $\frac{3}{5} \times \frac{1 - \frac{3}{5}\cos\theta}{1 - \frac{3}{5}\cos\theta} = \frac{3}{5} = e$* **or e.g.** $\frac{QF^2}{PF^2} = \frac{\left(3 - \frac{9}{5}\cos\theta\right)^2}{9\cos^2\theta - 30\cos\theta + 25} = \frac{9 - \frac{54}{5}\cos\theta + \frac{81}{25}\cos^2\theta}{9\cos^2\theta - 30\cos\theta + 25}$ $= \frac{9\left(1 - \frac{6}{5}\cos\theta + \frac{9}{25}\cos^2\theta\right)}{25\left(1 - \frac{6}{5}\cos\theta + \frac{9}{25}\cos^2\theta\right)}$ or e.g. $= \frac{9}{25} \times \frac{1 - \frac{6}{5}\cos\theta + \frac{9}{25}\cos^2\theta}{1 - \frac{6}{5}\cos\theta + \frac{9}{25}\cos^2\theta} = \frac{9}{25} \Rightarrow \frac{QF}{PF} = \frac{3}{5} = e$* Fully correct working including factorisation or equivalent leading to showing that $\frac{\lvert QF\rvert}{\lvert PF\rvert} = e$ with no errors and a conclusion "$= e$". Note that the value of e must have been seen earlier e.g. in part (b) or calculated independently somewhere in the question. Note that this mark depends on a ratio where the numerator and denominator are either both positive or both negative or modulus symbols are present throughout. This does not apply to the second case as both numerator and denominator must be positive as they are squared.		A1*
			(5)
			Total 12

Question Number	Scheme	Notes	Marks
1(a)	$1-\tanh^2 x \equiv \text{sech}^2 x$		
	$1-\tanh^2 x = 1-\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2$	Replaces the tanh x on the lhs with a **correct** expression in terms of exponentials.	B1
	$=\dfrac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}=\dfrac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}$ or e.g. $\dfrac{2e^{2x}\times 2e^{-2x}}{(e^x+e^{-x})^2}$ Attempts to find common denominator and expand numerator		M1
	$=\left(\dfrac{4}{(e^x+e^{-x})^2}\right)=\text{sech}^2 x$ *	Obtains the rhs with no errors.	A1cso
			(3)
ALT 1	$1-\tanh^2 x=(1-\tanh x)(1+\tanh x)$ $=\left(1-\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)\right)\left(1+\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)\right)$	Uses the difference of 2 squares on the lhs and replaces the tanh x with a **correct** expression in terms of exponentials.	B1
	$=\left(\dfrac{2e^{-x}}{e^x+e^{-x}}\right)\left(\dfrac{2e^x}{e^x+e^{-x}}\right)$	Attempt to find common denominators and simplify numerators.	M1
	$=\left(\dfrac{4}{(e^x+e^{-x})^2}\right)=\text{sech}^2 x$ *	Obtains the rhs with no errors.	A1cso
ALT 2	$\text{sech}^2 x=\dfrac{4}{(e^x+e^{-x})^2}$	Replaces the sech x on the rhs with a **correct** expression in terms of exponentials.	B1
	$=\dfrac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}=\dfrac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$ Attempts to express the "4" in terms of the denominator.		M1
	$=1-\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2=1-\tanh^2 x$ *	Obtains the lhs with no errors.	A1cso

(b)	$2\operatorname{sech}^2 x+3\tanh x=3\Rightarrow 2(1-\tanh^2 x)+3\tanh x=3$ $\Rightarrow 2\tanh^2 x-3\tanh x+1=0$ Uses $\operatorname{sech}^2 x=1-\tanh^2 x$ and forms a 3 term quadratic in $\tanh x$		M1
	$(2\tanh x-1)(\tanh x-1)=0\Rightarrow \tanh x=\ldots$	Solves 3TQ by any valid method including calculator.	M1
	$\tanh x=\frac{1}{2}\rightarrow x=\ln\sqrt{3}$	$\ln\sqrt{3}$. Accept $\frac{1}{2}\ln 3,\ -\frac{1}{2}\ln\frac{1}{3}$ And no other answers.	A1
			(3)
ALT	$2\operatorname{sech}^2 x+3\tanh x=3\Rightarrow 2\left(\frac{4}{(e^x+e^{-x})^2}\right)+3\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)=3$ $\Rightarrow 8+3(e^{2x}-e^{-2x})=3(e^{2x}+2+e^{-2x})\Rightarrow\ldots$ Substitutes the correct exponential forms, attempts to eliminate fractions and collect terms		M1
	$6e^{-2x}=2\Rightarrow e^{-2x}=\frac{1}{3}$	Rearranges to reach $e^{-2x}=\ldots$	M1
	$x=\ln\sqrt{3}$	$\ln\sqrt{3}$. Accept $\frac{1}{2}\ln 3,\ -\frac{1}{2}\ln\frac{1}{3}$ And no other answers.	A1
			Total 6

Question Number	Scheme	Notes	Marks
2.	$y=\sqrt{9-x^2}$, $0\le x\le 3$		
(a)	$\frac{dy}{dx}=-\frac{x}{\sqrt{9-x^2}}$	Correct derivative in any form.	B1
	Note that the derivative may be obtained implicitly after squaring e.g. $y=\sqrt{9-x^2}\Rightarrow y^2=9-x^2\Rightarrow 2y\frac{dy}{dx}=-2x\Rightarrow\frac{dy}{dx}=-\frac{x}{\sqrt{9-x^2}}$		
	Length of $C=\int\sqrt{1+\frac{x^2}{9-x^2}}\,dx$	Uses $\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ with their $\frac{dy}{dx}$	M1
	Note that the above may be obtained via the implicit route as e.g. $\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int\sqrt{1+\frac{x^2}{y^2}}\,dx=\int\sqrt{1+\frac{x^2}{9-x^2}}\,dx$ In which case the B1 is implied.		
	$=\int\sqrt{\frac{9}{9-x^2}}\,dx=3\arcsin\frac{x}{3}(+c)\ \left(\text{or } -3\arccos\frac{x}{3}(+c)\right)$ $\int_0^3\sqrt{\frac{9}{9-x^2}}\,dx=3\arcsin(1)-3\arcsin(0)\ \left(\text{or } -3\arccos(1)+3\arccos(0)\right)$ Finds common denominator, integrates to obtain arcsin... or arccos... and applies the limits 0 and 3.		M1
	$=\frac{3\pi}{2}$ *	Obtains the printed answer with no errors. This mark should be withheld if there is no evidence at all of the limits being applied.	A1
	<u>Special case:</u> If $+\frac{x}{\sqrt{9-x^2}}$ is obtained for $\frac{dy}{dx}$ score B0M1M1A1 if otherwise correct but allow full recovery in (b)		
			(4)
(b)	Surface Area $=\int 2\pi\sqrt{9-x^2}\left(\sqrt{\frac{9}{9-x^2}}\right)dx$	Uses $\int 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ with their $\frac{dy}{dx}$	M1
	$=\int_0^3 6\pi\,dx=6\pi\left[x\right]_0^3=\ldots$	Integrates to obtain kx and applies the limits 0 and 3. Condone omission of the lower limit.	M1
	$=18\pi$	18π cao	A1
			(3)
			Total 7

Question Number	Scheme	Notes	Marks
3.	$\mathbf{M} = \begin{pmatrix} 3 & 1 & p \\ 1 & 1 & 2 \\ -1 & p & 2 \end{pmatrix}$		
(a)	$\det \mathbf{M} = \begin{vmatrix} 3 & 1 & p \\ 1 & 1 & 2 \\ -1 & p & 2 \end{vmatrix}$ $= 3(2-2p) - 1(2+2) + p(p+1)$	Attempts determinant. Requires at least 2 correct "terms". May use other rows/columns or rule of Sarrus.	M1
	$= p^2 - 5p + 2$	Correct simplified determinant.	A1
	$p^2 - 5p + 2 = 0 \Rightarrow p = \ldots$	Solves 3TQ	M1
	$\frac{5 \pm \sqrt{17}}{2}$	Correct values.	A1
			(4)
(b)	Minors $\begin{pmatrix} 2-2p & 4 & p+1 \\ (2-p^2) & 6+p & (3p+1) \\ 2-p & (6-p) & 2 \end{pmatrix}$	Attempts the matrix of minors. If there is any doubt look for at least 6 correct elements. May be implied by their matrix of cofactors.	M1 **(B1 on EPEN)**
	Cofactors $\begin{pmatrix} 2-2p & -4 & p+1 \\ -(2-p^2) & 6+p & -(3p+1) \\ 2-p & -(6-p) & 2 \end{pmatrix}$	Attempts cofactors.	M1
		Correct matrix	A1
	$\mathbf{M}^{-1} = \frac{1}{p^2 - 5p + 2}\begin{pmatrix} 2-2p & p^2-2 & 2-p \\ -4 & 6+p & p-6 \\ p+1 & -3p-1 & 2 \end{pmatrix}$	Transposes matrix of cofactors and divides by determinant.	M1
		Follow though their det **M** from part (a) but the adjoint matrix must be correct.	A1ft
			(5)
			Total 9

Question Number	Scheme	Notes	Marks

4(i)	$f(x) = x\arccos x,\ -1 \le x \le 1,$		
	$f'(x) = \arccos x - \dfrac{x}{\sqrt{1-x^2}}$ M1: Differentiates using the product rule to obtain an expression of the form: $\arccos x \pm \dfrac{x}{\sqrt{1-x^2}}$ A1: Correct derivative		M1A1
	$f'(0.5) = \arccos 0.5 - \dfrac{0.5}{\sqrt{1-0.5^2}} = \dfrac{\pi-\sqrt{3}}{3}$	$\dfrac{\pi-\sqrt{3}}{3}$ oe e.g. $\dfrac{\pi}{3} - \dfrac{1}{\sqrt{3}}$	A1
			(3)
(ii)	$g(x) = \arctan(e^{2x})$		
	$g'(x) = \dfrac{2e^{2x}}{e^{4x}+1}$ M1: Differentiates using the chain rule to obtain an expression of the form: $\dfrac{ke^{2x}}{\left(e^{2x}\right)^2+1}$ A1: Correct derivative in any form		M1A1
	$g'(x) = \dfrac{2}{e^{2x}+e^{-2x}} = \text{sech}(2x)$	Introduces sech(2x). Depends on previous M.	**d**M1
	$g''(x) = -2\,\text{sech}(2x)\tanh(2x)$	Differentiates $\text{sech}(u) \to \pm\text{sech}\,u\tanh u$ Depends on both previous M's.	**d**M1
		Correct expression.	A1
			(5)
(ii) ALT 1	$g'(x) = \dfrac{2e^{2x}}{e^{4x}+1}$ M1: Differentiates using the chain rule to obtain an expression of the form: $\dfrac{ke^{2x}}{\left(e^{2x}\right)^2+1}$ A1: Correct derivative in any form		M1A1
	$g''(x) = \dfrac{4e^{2x}\left(1+e^{4x}\right) - 4e^{4x} \times 2e^{2x}}{(e^{4x}+1)^2}$	Differentiates using quotient or product rule. Depends on first M.	**d**M1
	$= \dfrac{4e^{2x} - 4e^{6x}}{(e^{4x}+1)^2} = \dfrac{-4(e^{2x}-e^{-2x})}{(e^{2x}+e^{-2x})^2}$	Multiply through by e^{-4x}. Depends on both previous M's.	**d**M1
	$= -2\dfrac{2}{e^{2x}+e^{-2x}}\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ $= -2\,\text{sech}\,2x\tanh 2x$	Correct expression.	A1
	Note that the first derivative may be obtained implicitly in either method e.g. $y = \arctan\left(e^{2x}\right) \Rightarrow \tan y = e^{2x} \Rightarrow \sec^2 y\dfrac{dy}{dx} = 2e^{2x} \Rightarrow \dfrac{dy}{dx} = \dfrac{2e^{2x}}{1+\left(e^{2x}\right)^2}$		
			Total 8

Question Number	Scheme	Notes	Marks
5.	$I_n = \int \sec^n x\,dx,\qquad n \ge 0$		

5(a)	$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$	Splits $\sec^n x$ into $\sec^{n-2} x \sec^2 x$	M1
	$\int \sec^n x\,dx = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\,dx$ **Depends on previous M mark** dM1: Uses integration by parts to obtain $\sec^{n-2} x \tan x - k\int \sec^{n-2} x \tan^2 x\,dx$ A1: Correct integration		**d**M1A1
	$\int \sec^n x\,dx = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x(\sec^2 x - 1)\,dx$ Uses $\tan^2 x = \sec^2 x - 1$		B1 **(M1 on EPEN)**
	$\int \sec^n x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$ $= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2} \Rightarrow (n-1)I_n = ...$ **Depends on all previous M and B marks** Introduces I_n and I_{n-2} and makes progress to the given result.		**dd**M1
	$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$ *	Fully correct proof.	A1cso
			(6)
ALT	$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$	Splits $\sec^n x$ into $\sec^{n-2} x \sec^2 x$	M1
	$\int \sec^{n-2} x \sec^2 x\,dx = \int \sec^{n-2} x\left(1 + \tan^2 x\right)dx$ $= \int \sec^{n-2} x\,dx + \int \tan^2 x \sec^{n-2} x\,dx$	Uses $\sec^2 x = 1 + \tan^2 x$ and splits into 2 integrals.	B1 **(4th mark M1 on EPEN)**
	$\int \tan^2 x \sec^{n-2} x\,dx = \frac{1}{(n-2)}\tan x \sec^{n-2} x - \frac{1}{(n-2)}\int \sec^n x\,dx$ Uses integration by parts on $\int \tan^2 x \sec^{n-2} x\,dx$ to obtain $A\tan x \sec^{n-2} x - B\int \sec^n x\,dx$ **Note this is the 2nd M on EPEN.**		**d**M1
	$\int \sec^n x\,dx = \int \sec^{n-2} x\,dx + \frac{1}{(n-2)}\tan x \sec^{n-2} x - \frac{1}{(n-2)}\int \sec^n x\,dx$ Fully correct integration		A1
	$\int \sec^n x\,dx = I_{n-2} + \frac{1}{(n-2)}\tan x \sec^{n-2} x - \frac{1}{(n-2)}I_n \Rightarrow (n-1)I_n = ...$ **Depends on previous M and B marks** Introduces I_n and I_{n-2} and makes progress to the given result.		**dd**M1
	$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$ *	Fully correct proof.	A1cso

5(b)	$I_2 = 1$	Correct value for I_2 seen or implied.	B1
	$I_6 = \frac{1}{5}\tan x \sec^4 x + \frac{4}{5}I_4$ or e.g.	Applies the given reduction formula once.	M1

	$I_6 = \frac{1}{5}\tan\frac{\pi}{4}\sec^4\frac{\pi}{4} + \frac{4}{5}I_4$ or e.g. $I_6 = \frac{1}{5}(1)\left(\sqrt{2}\right)^4 + \frac{4}{5}I_4$		
	$= \frac{1}{5}\tan x\sec^4 x + \frac{4}{5}\left(\frac{1}{3}\tan x\sec^2 x + \frac{2}{3}I_2\right) = \frac{1}{5}(1)\left(\sqrt{2}\right)^4 + \frac{4}{15}(1)\left(\sqrt{2}\right)^2 + \frac{8}{15}(1)$ Applies the given reduction formula again and uses the limits to reach a numerical expression for I_6		M1
	$= \frac{28}{15}$	Correct value	A1
			(4)
ALT	$I_2 = 1$	Correct value for I_2 seen or implied.	B1
	$I_4 = \frac{1}{3}\tan x\sec^2 x + \frac{2}{3}I_2$ or e.g. $I_4 = \frac{1}{3}\tan\frac{\pi}{4}\sec^2\frac{\pi}{4} + \frac{2}{3}I_2$ or e.g. $I_4 = \frac{1}{3}(1)\left(\sqrt{2}\right)^2 + \frac{2}{3}I_2$	Applies the given reduction formula once.	M1
	$I_6 = \frac{1}{5}\tan x\sec^4 x + \frac{4}{5}\left(\frac{1}{3}\tan x\sec^2 x + \frac{2}{3}I_2\right) = \frac{1}{5}(1)\left(\sqrt{2}\right)^4 + \frac{4}{15}(1)\left(\sqrt{2}\right)^2 + \frac{8}{15}$ Applies the given reduction formula again and uses the limits to reach a numerical expression for I_6		M1
	$= \frac{28}{15}$	Correct value	A1
			Total 10

In part (b), condone confusion with the coefficients provided the intention is clear.

For either method in part (b), all working must be shown and the given reduction formula must be used at least once. So do not allow e.g. I_4 to be evaluated with a calculator but I_4 can be evaluated directly without using the given reduction formula using an alternative method e.g. by parts or by substitution – see below:

Parts:

$$I_4 = \int \sec^4 x\,dx = \int \sec^2 x\sec^2 x\,dx = \sec^2 x\tan x - 2\int \sec^2 x\tan^2 x\,dx$$

$$= \sec^2 x\tan x - 2\int \sec^2 x\left(\sec^2 x - 1\right)dx = \sec^2 x\tan x - 2\int \sec^4 x\,dx + 2\int \sec^2 x\,dx$$

$$= \sec^2 x\tan x - 2I_4 + 2\int \sec^2 x\,dx \Rightarrow 3I_4 = \sec^2 x\tan x + 2\tan x \Rightarrow I_4 = \frac{1}{3}\sec^2 x\tan x + \frac{2}{3}\tan x$$

Substitution:

$$I_4 = \int \sec^4 x\,dx = \int \sec^2 x\sec^2 x\,dx = \int \sec^2 x\left(1+\tan^2 x\right)dx$$

$$u = \tan x \Rightarrow \int \sec^2 x\left(1+\tan^2 x\right)dx = \int \sec^2 x\left(1+u^2\right)\frac{du}{\sec^2 x} = \frac{u^3}{3} + u = \frac{\tan^3 x}{3} + \tan x$$

Question Number	Scheme	Notes	Marks

6(a)	Normal to plane given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \ldots$	Attempt cross product of direction vectors. If the method is unclear, look for at least 2 correct components.	M1
	$= 6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$	Or any multiple of this vector.	A1
	Substitute appropriate point into $6x + 2y - 2z = d$ e.g. (1, 1, 1) or (2, 1, 4) to find "*d*"	Use a valid point and use scalar product with normal or substitute into Cartesian equation.	M1
	$6x + 2y - 2z = 6$ $3x + y - z = 3$ *	Given answer. No errors seen	A1* cso
			(4)
6(a) ALT	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ $\Rightarrow x = 1 + \lambda + \mu,\ y = 1 - 2\mu,\ z = 1 + 3\lambda + \mu$ M1: Forms equation of plane using (1, 1, 1) and direction vectors and extracts 3 equations for *x*, *y* and *z* in terms of λ and μ A1: Correct equations		M1A1
	$x = 1 + \frac{1}{2} - \frac{1}{2}y + \frac{1}{3}z - \frac{1}{2} + \frac{1}{6}y$	Eliminates λ and μ and achieves an equation in *x*, *y* and *z* only.	M1
	$3x + y - z = 3$ *	Given answer. No errors seen.	A1
6(b)	$s = -3$	cao	B1
			(1)
6(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & 1 & -1 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$	Attempts cross product of normal vectors. If the method is unclear, look for at least 2 correct components.	M1
	e.g. $x = 0, 2y - 2z = 6, y - 2z = 3$ $\Rightarrow y = 3, z = 0$	Any valid attempt to find a point on the line.	M1
	e.g. (0,3,0)	Any valid point on the line	A1
	$\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$	Correct equation including "$\mathbf{r}$ =" or equivalent e.g. $x = \frac{y-3}{-5} = \frac{z}{-2}$	A1
			(4)
6(c) ALT 1	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}),\ \mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 3$ $\Rightarrow 1 + \lambda + \mu + 1 - 2\mu - 2 - 6\lambda - 2\mu = 3$ Forms equation of first plane using (1, 1, 1) and direction vectors and substitutes into the second plane to form an equation in λ and μ		M1
	$\Rightarrow \mu = \frac{1}{3}(-5\lambda - 3)$	Solves to obtain μ in terms of λ or λ in terms of μ	M1
		Correct equation	A1
	E.g. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \frac{1}{3}(-5\lambda - 3)(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ Correct equation including "$\mathbf{r}$ ="		A1
6(c) ALT 2	$3x + y - z = 3,\ x + y - 2z = 3 \Rightarrow 2x + z = 0$	Uses the Cartesian equations of both planes and eliminates one variable	M1
	$z = \lambda \Rightarrow x = -\frac{1}{2}\lambda,\ y = 3 + 2z - x = 3 + \frac{5}{2}\lambda$	Introduces parameter and expresses other 2 variables in terms of the parameter	M1
		Correct equations	A1
	$\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$	Correct equation including "$\mathbf{r}$ =" or equivalent e.g. $x = \frac{y-3}{-5} = \frac{z}{-2}$	A1

6(c) ALT 3	$3x + y - z = 3,\ x + y - 2z = 3 \Rightarrow 2x + z = 0$	Uses the Cartesian equations of both planes and eliminates one variable	M1

	$3x+y-z=3,\ x+y-2z=3 \Rightarrow 5x+y=3$	Uses the Cartesian equations of both planes and eliminates another variable	M1
	$\Rightarrow x=-\frac{z}{2},\quad x=\frac{3-y}{5}$	Correct equations for one variable in terms of the other 2	A1
	$x=\frac{y-3}{-5}=\frac{z}{-2}$	Correct equation or equivalent e.g. $x=\frac{3-y}{5}=\frac{z}{-2}$	A1
6(d)	$(3\mathbf{i}+\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-2\mathbf{k})=6$	Correct value for scalar product	B1
	$\cos\theta=\frac{(3\mathbf{i}+\mathbf{j}-\mathbf{k}).(\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{9+1+1}\sqrt{1+1+4}}=\sqrt{\frac{6}{11}}$	Full scalar product attempt to reach a value for $\cos\theta$	M1
		For $\cos\theta=\sqrt{\frac{6}{11}}$	A1
	$\theta=42.4°$	Correct value. Mark their final answer.	A1
			(4)
6(d) ALT	$\lvert(3\mathbf{i}+\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{j}-2\mathbf{k})\rvert=\sqrt{30}$	Correct value for magnitude of cross product	B1
	$\sin\theta=\frac{\lvert(3\mathbf{i}+\mathbf{j}-\mathbf{k}).(\mathbf{i}+\mathbf{j}-2\mathbf{k})\rvert}{\sqrt{9+1+1}\sqrt{1+1+4}}=\frac{\sqrt{55}}{11}$	Full attempt to reach a value for $\sin\theta$	M1
		For $\sin\theta=\frac{\sqrt{55}}{11}$	A1
	$\theta=42.4°$	Correct value. Mark their final answer.	A1
			Total 13

Question Number	Scheme	Notes	Marks
7(i)	$x^2 - 4x + 5 = (x-2)^2 + 1$	Attempts to complete the square. Allow for $(x-2)^2 + c,\ c > 0$	M1
	$\int \frac{1}{(x-2)^2+1} dx = \arctan(x-2)$	Allow for $k\arctan f(x)$.	M1
	$[\arctan(x-2)]_1^2 = 0 - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$	$\frac{\pi}{4}$ cao	A1
			(3)
7(ii)	$\int \frac{\sqrt{x^2-3}}{x^2} dx = -\frac{\sqrt{x^2-3}}{x} + \int \frac{1}{\sqrt{x^2-3}} dx$ Uses integration by parts and obtains $A\frac{\sqrt{x^2-3}}{x} + B\int \frac{1}{\sqrt{x^2-3}} dx$		M1
	$= -\frac{\sqrt{x^2-3}}{x} + \text{arcosh}\frac{x}{\sqrt{3}}$	$B\int \frac{1}{\sqrt{x^2-3}} dx = k\text{arcosh } f(x)$	M1
		All correct	A1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2} dx = \left[-\frac{\sqrt{x^2-3}}{x} + \text{arcosh}\frac{x}{\sqrt{3}}\right]_{\sqrt{3}}^{3} = \left(-\frac{\sqrt{6}}{3} + \text{arcosh}\sqrt{3}\right) - (0 + \text{arcosh}1)$ Applies the limits 3 and $\sqrt{3}$ **Depends on both previous M marks**		**d**M1
	$\text{arcosh}\sqrt{3} - \frac{1}{3}\sqrt{6} = \ln(\sqrt{2}+\sqrt{3}) - \frac{1}{3}\sqrt{6}$	Accept either of these forms.	A1
			(5)
7(ii) ALT 1	$\int \frac{\sqrt{x^2-3}}{x^2} dx = \int \frac{\sqrt{3\cosh^2 u - 3}}{3\cosh^2 u}\sqrt{3}\sinh u\, du$	A complete substitution using $x = \sqrt{3}\cosh u$	M1
	$= \int \tanh^2 u\, du$	Obtains $k\int \tanh^2 u\, du$	M1
	$= \int (1 - \text{sech}^2 u) du = u - \tanh u$	Correct integration	A1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2} dx = [u - \tanh u]_0^{\text{arcosh}\sqrt{3}} = \text{arcosh}\sqrt{3} - \tanh\left(\text{arcosh}\sqrt{3}\right) - 0$ Applies the limits 0 and arcosh$\sqrt{3}$ **Depends on both previous M marks**		**d**M1
	$\text{arcosh}\sqrt{3} - \frac{1}{3}\sqrt{6} = \ln(\sqrt{2}+\sqrt{3}) - \frac{1}{3}\sqrt{6}$	Accept either of these forms.	A1

7(ii) ALT 2	$\int \frac{\sqrt{x^2-3}}{x^2}\,dx = \int \frac{\sqrt{3\sec^2 u-3}}{3\sec^2 u}\sqrt{3}\sec u\tan u\,du$	A complete substitution using $x = \sqrt{3}\sec u$	M1
	$= \int \frac{\tan^2 u}{\sec u}\,du$	Obtains $k\int \frac{\tan^2 u}{\sec u}\,du$	M1
	$= \ln(\sec u + \tan u) - \sin u$	Correct integration	A1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2}\,dx = [\ln(\sec u + \tan u) - \sin u]_0^{\operatorname{arcsec}\sqrt{3}}$ $= \ln\left(\sec\left(\operatorname{arcsec}\sqrt{3}\right) + \tan\left(\operatorname{arcsec}\sqrt{3}\right)\right) - \ln\left(\sec(0)+\tan(0)\right) - \sin\left(\operatorname{arcsec}\sqrt{3}\right)$ Applies the limits 0 and arcsec√3 **Depends on both previous M marks**		**d**M1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2}\,dx = \ln(\sqrt{2}+\sqrt{3}) - \frac{1}{3}\sqrt{6}$	Correct answer.	A1
			Total 8

Note that there may be other ways to perform the integration in part (ii) e.g. subsequent substitutions. Marks can be awarded if the method leads to something that is integrable and should be awarded as in the main scheme e.g. M1 for a complete method, M2 for simplifying and reaching an expression that itself can be integrated or can be integrated after rearrangement, A1 for correct integration, dM3 for using appropriate limits and A2 as above.

Alternative approach:

$$\int \frac{\sqrt{x^2-3}}{x^2}\,dx = \int \frac{x^2-3}{x^2\sqrt{x^2-3}}\,dx = \int \frac{1}{\sqrt{x^2-3}}\,dx - \int \frac{3}{x^2\sqrt{x^2-3}}\,dx = \operatorname{arcosh}\frac{x}{\sqrt{3}} - \ldots$$

Can score **M0M1A0dM0A0** if there is no creditable attempt at the second integral.

If the second integral is attempted, it must be using a suitable method
e.g. with either $x = \sqrt{3}\cosh u$ or $x = \sqrt{3}\sec u$:

$$\int \frac{3}{x^2\sqrt{x^2-3}}\,dx = \int \frac{3}{3\cosh^2 u\sqrt{3\cosh^2 u - 3}}\sqrt{3}\sinh u\,du = \int \operatorname{sech}^2 u\;du = \tanh u + c$$

or

$$\int \frac{3}{x^2\sqrt{x^2-3}}\,dx = \int \frac{3}{3\sec^2 u\sqrt{3\sec^2 u - 3}}\sqrt{3}\sec u\tan u\,du = \int \cos u\;du = \sin u + c$$

In these cases the first M can then be awarded and the other marks as defined with the appropriate limits used.

Question Number	Scheme	Notes	Marks
8(a)	Asymptotes are $y = \pm 2x$	$y = \pm 2x$ oe e.g. $x = \pm \frac{y}{2}$	B1
			(1)
8(b)	$4 = e^2 - 1 \Rightarrow e = \sqrt{5}$	Uses the correct eccentricity formula with a = 1 and b = 2 to find a value for e.	M1
	Foci are $(\pm\sqrt{5}, 0)$	Both required.	A1
			(2)
8(c)	$8x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{4x}{y} = \frac{4\sec\theta}{2\tan\theta}$ or $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta}$ M1: $Ax + By\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = f(\theta)$ or $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = f(\theta)$ A1: Correct gradient in terms of θ		M1A1
	Explicit differentiation may be seen: $y^2 = 4x^2 - 4 \Rightarrow y = (4x^2 - 4)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(4x^2 - 4)^{-\frac{1}{2}} \times 8x = \frac{4\sec\theta}{\sqrt{4\sec^2\theta - 4}}$ Score M1 for $\frac{dy}{dx} = kx(4x^2 - 4)^{-\frac{1}{2}} = f(\theta)$ and A1 for correct gradient in terms of θ		
	E.g. $y - 2\tan\theta = \frac{4\sec\theta}{2\tan\theta}(x - \sec\theta)$	Correct straight line method using their gradient in terms of θ and x = sec θ, y = 2tan θ	M1
	$y\tan\theta - 2\tan^2\theta = 2x\sec\theta - 2\sec^2\theta$ $\Rightarrow y\tan\theta - 2\tan^2\theta = 2x\sec\theta - 2(1 + \tan^2\theta)$		
	$y\tan\theta = 2x\sec\theta - 2$ *	Obtains the given answer with sufficient working shown as above.	A1cso
			(4)
8(d)	$VP : V(-1, 0); P(\sec\theta, 2\tan\theta) \Rightarrow y = \frac{2\tan\theta}{\sec\theta + 1}(x + 1)$ or $WQ : W(1, 0); Q(\sec\theta. -2\tan\theta) \Rightarrow y = \frac{-2\tan\theta}{\sec\theta - 1}(x - 1)$ M1: Correct straight line method for either *VP* or *WQ* A1: One correct equation in any form		M1A1
	$y = \frac{-2\tan\theta}{\sec\theta - 1}(x - 1),\ y = \frac{2\tan\theta}{\sec\theta + 1}(x + 1)$	Both equations correct in any form.	A1
	$\frac{2\tan\theta}{\sec\theta + 1}(x + 1) = \frac{-2\tan\theta}{\sec\theta - 1}(x - 1) \Rightarrow x / y = ...$	Attempt to solve and makes progress to achieve either x = … or y = …in terms of θ only.	M1
	$x = \cos\theta$ or $y = 2\sin\theta$	One correct coordinate	A1
	$x = \cos\theta$ and $y = 2\sin\theta$	Both correct	A1
	$x^2 + \frac{y^2}{4} = 1$ or $a = 1, b = 2$	Correct equation or correct values for a and b	A1
			(7)
			Total 14

Question	Scheme	Marks
1	$\dfrac{dy}{dx}=\dfrac{1}{2}\times\dfrac{2}{\sqrt{(2x)^2-1}}$	M1
	$1+\left(\dfrac{dy}{dx}\right)^2=1+\dfrac{1}{4x^2-1}=\dfrac{4x^2}{4x^2-1}$	M1
	$\displaystyle\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int\sqrt{\frac{4x^2}{4x^2-1}}\,dx=2\int\frac{x}{\sqrt{4x^2-1}}\,dx$	A1
	$=\dfrac{2\left(4x^2-1\right)^{\frac{1}{2}}}{8\times{}^1\!/_2}$	M1
	$s=\left[\dfrac{\left(4x^2-1\right)^{\frac{1}{2}}}{2}\right]_{\frac{7}{2}}^{13}=\dfrac{1}{2}\left(\sqrt{4\times169-1}-\sqrt{4\times\dfrac{49}{4}-1}\right)=...$	dM1
	$=\dfrac{1}{2}\left(15\sqrt{3}-4\sqrt{3}\right)=\dfrac{11}{2}\sqrt{3}$	A1
		(6)
		(6 marks)

Notes:

M1: Attempts $\dfrac{dy}{dx}$, accept the form $\dfrac{A}{\sqrt{(2x)^2-1}}$. Allow $\dfrac{A}{\sqrt{2x^2-1}}$ (condone missing brackets)

Alternative 1:

Writes $\dfrac{1}{2}\operatorname{ar}\cosh 2x$ as $\dfrac{1}{2}\ln\left(2x+\sqrt{4x^2-1}\right)$ leading to

$$\frac{dy}{dx}=\frac{1}{2}\times\frac{1}{2x+\sqrt{4x^2-1}}\times\left(2+\frac{4x}{\sqrt{4x^2-1}}\right)=\frac{2x+\sqrt{4x^2-1}}{\sqrt{4x^2-1}\left(2x+\sqrt{4x^2-1}\right)}=\frac{1}{\sqrt{4x^2-1}}$$

Alternative 2:

$$y=\frac{1}{2}\operatorname{ar}\cosh 2x\Rightarrow 2y=\operatorname{ar}\cosh 2x\Rightarrow\cosh 2y=2x\rightarrow 4\sinh 2y\frac{dy}{dx}=2x\Rightarrow\frac{dy}{dx}=\frac{1}{\sinh 2y}=\frac{1}{\sqrt{4x^2-1}}$$

If either approach is taken then the same condition for the form of the derivative applies.

Note that this differentiation may be seen in an attempt by parts of $\int y\,dx$

M1: Attempts to find $1+\left(\dfrac{dy}{dx}\right)^2$ using their $\dfrac{dy}{dx}$ and attempts common denominator.

A1: Reaches a correct simplified integral with $\sqrt{x^2}$ replaced with x as shown in the scheme.

Allow equivalent forms e.g. $2\displaystyle\int x\sqrt{\frac{1}{4x^2-1}}dx$, $\dfrac{1}{2}\displaystyle\int\frac{4x}{\sqrt{(2x)^2-1}}dx$

This may be implied by subsequent work.

M1: Attempts the integration and reaches the form $\alpha\left(\beta x^2-1\right)^{\frac{1}{2}}$. α and/or β may be 1

This may be implied by e.g.

$$u=4x^2-1 \to k\int \frac{1}{\sqrt{u}}\,\mathrm{d}u=\alpha\sqrt{u} \quad \text{or} \quad u=x^2 \to k\int \frac{1}{\sqrt{4u-1}}\,\mathrm{d}u=\alpha\sqrt{4u-1}$$

dM1: Applies the limits to their integral. **Depends on the previous 2 method marks.**
Any attempts at substitution requires use of changed limits e.g.

$$u=4x^2-1 \to \frac{1}{4}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u \to \frac{1}{2}\left[\sqrt{u}\right]_{48}^{675}=\ldots$$

A1: cao Accept equivalents in the correct form, such as $\frac{1}{2}\sqrt{363}$

Examples of alternative for the final 3 marks:

$$x=\frac{1}{2}\cosh u \Rightarrow 2\int \frac{x}{\sqrt{4x^2-1}}\,\mathrm{d}x=\int \frac{\cosh u}{\sqrt{\cosh^2 u-1}}\frac{1}{2}\sinh u\,\mathrm{d}u$$

$$\int \frac{1}{2}\cosh u\,\mathrm{d}u=\frac{1}{2}\left[\sinh u\right]_{\text{arcosh}\,7}^{\text{arcosh}\,26}=\frac{1}{2}\left(\frac{e^{\ln(26+15\sqrt{3})}-e^{-\ln(26+15\sqrt{3})}}{2}-\frac{e^{\ln(7+4\sqrt{3})}-e^{-\ln(7+4\sqrt{3})}}{2}\right)$$

$$=\frac{1}{2}\left(15\sqrt{3}-4\sqrt{3}\right)=\frac{11}{2}\sqrt{3}$$

Score M1 for a complete method for the substitution leading to $k\sinh u$ and then dM1 for applying changed limits (or reverts back to x) and A1 as above

$$x=\frac{1}{2}\sec u \Rightarrow 2\int \frac{x}{\sqrt{4x^2-1}}\,\mathrm{d}x=\int \frac{\sec u}{\sqrt{\sec^2 u-1}}\frac{1}{2}\sec u\tan u\,\mathrm{d}u$$

$$\int \frac{1}{2}\sec^2 u\,\mathrm{d}u=\frac{1}{2}\left[\tan u\right]_{\text{arcosh}\frac{1}{7}}^{\text{arcosh}\frac{1}{26}}$$

$$=\frac{1}{2}\left(15\sqrt{3}-4\sqrt{3}\right)=\frac{11}{2}\sqrt{3}$$

Score M1 for a complete method for the substitution leading to $k\tan u$ and then dM1 for applying changed limits (or reverts back to x) and A1 as above

Special Case if no integration is attempted:
Note that if candidates do not attempt the integration but obtain the correct exact answer then a special case of **M1M1A1M0A0A1** (4/6) should be awarded.

Question	Scheme	Marks
2.	$\cosh y = x, y < 0 \Rightarrow y = \ln\left[x - \sqrt{x^2 - 1}\right]$	
	$\cosh y = x \Rightarrow x = \dfrac{e^y + e^{-y}}{2}$	**B1**
	$\Rightarrow 2xe^y = e^{2y} + 1$	**M1**
	$\Rightarrow e^{2y} - 2xe^y + 1 = 0 \Rightarrow e^y = \dfrac{2x \pm \sqrt{(2x)^2 - 4 \times 1 \times 1}}{2}$ or $\Rightarrow e^{2y} - 2xe^y + 1 = 0 \Rightarrow \left(e^y - x\right)^2 + 1 - x^2 = 0 \Rightarrow e^y = \ldots$	**M1**
	$= x \pm \sqrt{x^2 - 1}$	**A1**
	So $y = \ln\left[x - \sqrt{x^2 - 1}\right]$ *	**A1***
	since $y < 0 \Rightarrow e^y < 1$ so need $x - \sqrt{x^2 - 1}$ (as $x > 1$ so must subtract)	**B1**
		(6)
		(6 marks)

Notes:

B1: Correct statement for x in terms of exponentials. $\cosh y = \dfrac{e^x + e^{-x}}{2}$ scores B0.

M1: Multiplies through by e^y to achieve a quadratic in e^y. (Terms need not be gathered.)
M1: Uses the quadratic formula or other valid method (e.g. completing the square) to solve for e^y.

A1: Correct solution(s) for e^y. Accept if only the negative one is given. Accept $\dfrac{2x \pm \sqrt{4x^2 - 4}}{2}$

A1*: Completely correct work leading to the given answer regardless of the justification why the negative root is taken (correct or incorrect). Must be no errors seen.
B1: Suitable justification for taking the negative root given.
E.g. $y < 0$ so $y = \ln\left[x - \sqrt{x^2 - 1}\right]$. Condone $x \pm \sqrt{x^2 - 1} < 1$ so $y = \ln\left[x - \sqrt{x^2 - 1}\right]$.

Note that the B1 can only be awarded if all previous marks have been awarded.
But the reason may be given before or after ln has been taken.
E.g. $\left(e^y - x\right)^2 + 1 - x^2 = 0 \Rightarrow e^y - x = \pm\sqrt{x^2 - 1}$ but $y < 0$ so $e^y - x = -\sqrt{x^2 - 1}$

Working backwards:

$$y = \ln\left[x - \sqrt{x^2 - 1}\right] \Rightarrow e^y = x - \sqrt{x^2 - 1}\,(\text{B1}) \Rightarrow e^y + e^{-y} = x - \sqrt{x^2 - 1} + \frac{1}{x - \sqrt{x^2 - 1}}\,(\text{M1})$$

$$x - \sqrt{x^2 - 1} + \frac{1}{x - \sqrt{x^2 - 1}} = \frac{2x\left(x - \sqrt{x^2 - 1}\right)}{x - \sqrt{x^2 - 1}}\,(\text{M1}) = 2x\,(\text{A1}) \Rightarrow x = \frac{e^y + e^{-y}}{2} = \cosh y\,(\text{A1})$$

Final B1 unlikely to be available.

Question	Scheme	Marks
3(a)	$\frac{dy}{dx}=\frac{6\cos\theta}{-8\sin\theta}$ or $\frac{2x}{64}+\frac{2y}{36}\frac{dy}{dx}=0$ or $\frac{dy}{dx}=\frac{1}{4}\times\frac{1}{2}\left(576-9x^2\right)^{-\frac{1}{2}}\times -18x$	B1
	$m_T=-\frac{3\cos\theta}{4\sin\theta}\Rightarrow m_N=-\frac{1}{m_T}=\frac{4\sin\theta}{3\cos\theta}$	M1
	So normal is $y-6\sin\theta=\frac{4\sin\theta}{3\cos\theta}(x-8\cos\theta)$ or $y=\frac{4\sin\theta}{3\cos\theta}x+c,\ c=6\sin\theta-\frac{4\sin\theta}{3\cos\theta}\times 8\cos\theta$	dM1
	$\Rightarrow 3y\cos\theta-18\sin\theta\cos\theta=4x\sin\theta-32\sin\theta\cos\theta$ $\Rightarrow 4x\sin\theta-3y\cos\theta=14\sin\theta\cos\theta$ *	A1*
		(4)
(b)	A is $\left(\frac{7}{2}\cos\theta,0\right)$ and B is $\left(0,-\frac{14}{3}\sin\theta\right)$	B1
	M is $\left(\frac{\frac{7}{2}\cos\theta}{2},-\frac{\frac{14}{3}\sin\theta}{2}\right)=\left(\frac{7}{4}\cos\theta,-\frac{7}{3}\sin\theta\right)$	M1
	$\sin^2\theta+\cos^2\theta=1\Rightarrow\left(-\frac{3}{7}y\right)^2+\left(\frac{4}{7}x\right)^2=1$	dM1 A1
	$\Rightarrow 16x^2+9y^2=49$	A1
		(5)
		(9 marks)

Notes:

(a)

B1: A correct statement for, or involving, $\frac{dy}{dx}$. See examples in scheme for parametric, implicit and direct forms.

M1: Finds $\frac{dy}{dx}$ in terms of θ and applies the perpendicular condition to find gradient of the normal.

dM1: Uses their normal gradient and P to find the equation of the normal

A1*: Correct answer from correct work with at least one intermediate step and no errors seen.

(b)

B1: Correct coordinates for A and B or correct intercepts of l seen or implied by working. Allow in any form simplified or unsimplified.

M1: Uses their A and B to attempt the midpoint, M. May be implied by at least one correct coordinate.

dM1: Uses $\sin^2\theta+\cos^2\theta=1$ with their M to form an equation in x and y only.

Depends on the previous mark.

A1: A correct unsimplified equation.

A1: Correct equation in the required form. Allow any integer multiple.

Special Case: If M is found as e.g. $\left(\frac{7}{4}\cos\theta, \frac{7}{3}\sin\theta\right)$ withhold the final mark only if otherwise correct.

Question	Scheme	Marks
4(a)	$\begin{vmatrix} 2 & 0 & -1 \\ k & 3 & 2 \\ -2 & 1 & k \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 1 & k \end{vmatrix} - 0\begin{vmatrix} k & 2 \\ -2 & k \end{vmatrix} + (-1)\begin{vmatrix} k & 3 \\ -2 & 1 \end{vmatrix} = 2(3k-2)-(k+6) = \ldots$	M1
	$= 6k-4-k-6 = 5k-10$ *	A1*
		(2)
(b)	$\mathbf{M}^T = \begin{pmatrix} 2 & k & -2 \\ 0 & 3 & 1 \\ -1 & 2 & k \end{pmatrix}$ or minors $\begin{pmatrix} 3k-2 & k^2+4 & k+6 \\ 1 & 2k-2 & 2 \\ 3 & 4+k & 6 \end{pmatrix}$ or cofactors $\begin{pmatrix} 3k-2 & -k^2-4 & k+6 \\ -1 & 2k-2 & -2 \\ 3 & -4-k & 6 \end{pmatrix}$	M1
	Adjugate matrix is $\begin{pmatrix} 3k-2 & -1 & 3 \\ -k^2-4 & 2k-2 & -4-k \\ k+6 & -2 & 6 \end{pmatrix}$ (≥ 6 entries correct)	M1
	Hence $\mathbf{M}^{-1} = \frac{1}{5k-10}\begin{pmatrix} 3k-2 & -1 & 3 \\ -k^2-4 & 2k-2 & -4-k \\ k+6 & -2 & 6 \end{pmatrix}$	dM1A1
		(4)
(c)	Images of A, B and C are $(5, 4k-18, 3k-16)$, $(0, 7-2k, 9-4k)$ and $(0, 4k-2, 8k-14)$	M1 A1
	$(\pm)50 = \frac{1}{6}\begin{vmatrix} 5 & 4k-18 & 3k-16 \\ 0 & 7-2k & 9-4k \\ 0 & 4k-2 & 8k-14 \end{vmatrix} \Rightarrow (\pm)300 = 5(\ldots)(= 200k-400) \Rightarrow k = \ldots$	M1
	$(300 = 200k-400 \Rightarrow) k = \frac{7}{2}$ or $(-300 = 200k-400 \Rightarrow) k = \frac{1}{2}$	A1
	$k = \frac{1}{2}$ and $k = \frac{7}{2}$	A1
		(5)
Alt method	Using volume scale factor. Attempts $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 4 & -8 & 3 \\ -2 & 5 & -4 \\ 4 & -6 & 8 \end{vmatrix} = 4(40-24)+8(-16+16)+3(12-20) = \ldots$	M1
	Volume of T is $\frac{1}{6}\lvert\mathbf{a}.(\mathbf{b}\times\mathbf{c})\rvert = \left\lvert\frac{1}{6}\begin{vmatrix} 4 & -8 & 3 \\ -2 & 5 & -3 \\ 4 & 6 & -8 \end{vmatrix}\right\rvert = \ldots \frac{20}{3}$	A1
	Volume image of $T = \lvert\det \mathbf{M}\rvert \times \frac{20}{3} \Rightarrow \frac{20}{3}\lvert 5k-10\rvert = 50 \Rightarrow k = \ldots$	M1

	$\left(\frac{20}{3}(5k-10)=50 \Rightarrow\right) k=\frac{7}{2}$ or $\left(\frac{20}{3}(10-5k)=50 \Rightarrow\right) k=\frac{1}{2}$	**A1**
	$k=\frac{1}{2}$ and $k=\frac{7}{2}$	**A1**
		(5)
		(11 marks)

Notes:

(a)

M1: Correct method for expanding the determinant to reach a linear expression in k. Expect expansion along the top row, but may expand along any row or column. Sarrus gives $6 + k - (6 + 4)$.

A1*: Correct expression from correct work.

(b)

M1: Begins the process of finding the inverse by attempting either the transpose, or the matrix of minors or cofactors. Look for at least 6 correct entries.

M1: Proceeds to find the adjugate matrix (may include the reciprocal determinant). Again look for 6 correct entries.

dM1: Full method to find the inverse matrix, so divides their adjugate by the determinant. **Depends on both previous marks.**

A1: Fully correct inverse.

(c)

M1: Attempts to find the image vectors of A, B and C under the transformation. (O mapping to O may be assumed). May be implied by at least two correct entries in one of the three vectors – but must be finding all three.

A1: Correct image vectors. Allow unsimplified and isw if necessary.

M1: Use their image vectors in a suitable scalar triple product to find the volume, and set volume equal to 50 and attempts to solve for k. Must include the 1/6 but may appear later.

$$\text{Usually } \frac{1}{6}(200k-400)=50 \text{ leading to } k=\frac{7}{2}$$

A1: One correct value for k obtained, either $k=\frac{7}{2}$ or $k=\frac{1}{2}$

A1: Both values of k correctly found. $k=\frac{7}{2}$ and $k=\frac{1}{2}$

Alt method using determinant as volume scale factor.

M1: Attempts an appropriate scalar triple product. May have rows in different order.

A1: Correct volume for tetrahedron T. Need not be simplified, so $\frac{40}{6}$ is fine here.

M1: Uses the determinant as the volume scale factor to set up at least one equation in k using their volume and the given volume and attempts to solve for k. The 1/6 may have been missing.

$$\text{Usually } \frac{20}{3}(5k-10)=50 \text{ leading to } k=\frac{7}{2}$$

A1: One correct value for k obtained, either $k=\frac{7}{2}$ or $k=\frac{1}{2}$

A1: Both values of k correctly found. $k=\frac{7}{2}$ and $k=\frac{1}{2}$

Question	Scheme	Marks
5(a)	$(5\mathbf{i}+\mathbf{j})\times(8\mathbf{i}-2\mathbf{j}+3\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & 1 & 0\\ 8 & -2 & 3\end{vmatrix}=...$ Or $\left.\begin{matrix}(u\mathbf{i}+v\mathbf{j}+w\mathbf{k}).(5\mathbf{i}+\mathbf{j})=0\\ (u\mathbf{i}+v\mathbf{j}+w\mathbf{k}).(8\mathbf{i}-2\mathbf{j}+3\mathbf{k})=0\end{matrix}\right\}\Rightarrow \begin{matrix}5u+v=0\\ 8u-2v+3w=0\end{matrix}\Rightarrow u,v,w=...$	M1
	$\mathbf{n}=3\mathbf{i}-15\mathbf{j}-18\mathbf{k}$ or $\alpha(\mathbf{i}-5\mathbf{j}-6\mathbf{k})$ for any $\alpha\neq 0$	A1
		(2)
(b)	(i) $\mathbf{r}=(2\mathbf{i}-4\mathbf{j}+4\mathbf{k})+s(8\mathbf{i}-2\mathbf{j}+3\mathbf{k})+t(5\mathbf{i}+\mathbf{j})$	B1
		(1)
	(ii) $(2\mathbf{i}-4\mathbf{j}+4\mathbf{k}).(3\mathbf{i}-15\mathbf{j}-18\mathbf{k})=...(=-6)$	M1
	So $\mathbf{r}.(3\mathbf{i}-15\mathbf{j}-18\mathbf{k})=-6$ oe such as $\mathbf{r}.(-\mathbf{i}+5\mathbf{j}+6\mathbf{k})=2$	A1
		(2)
(c) Way 1	Distance from plane in (b) to origin is $\dfrac{\pm 6}{\sqrt{3^2+15^2+18^2}}$ oe e.g. $\dfrac{2}{\sqrt{1^2+5^2+6^2}}$ Or attempts similar for parallel plane containing l_1, e.g. $\dfrac{(\mathbf{i}+2\mathbf{j}-5\mathbf{k}).(3\mathbf{i}-15\mathbf{j}-18\mathbf{k})}{\sqrt{3^2+15^2+18^2}}=...$	M1
	$=\pm\dfrac{2}{\sqrt{62}}$ (oe evaluated) or $\mp\dfrac{21}{\sqrt{62}}$ if considering other plane.	A1
	Both $\dfrac{\pm 6}{\sqrt{3^2+15^2+18^2}}$ oe and $\dfrac{(\mathbf{i}+2\mathbf{j}-5\mathbf{k}).(3\mathbf{i}-15\mathbf{j}-18\mathbf{k})}{\sqrt{3^2+15^2+18^2}}=...$ attempted	M1
	Hence shortest distance between lines is $\dfrac{2}{\sqrt{62}}+\dfrac{21}{\sqrt{62}}=...$	M1
	$=\dfrac{23}{\sqrt{62}}$ or $\dfrac{23\sqrt{62}}{62}$	A1
		(5)
Way 2	$\overrightarrow{AB}=\pm((\mathbf{i}+2\mathbf{j}-5\mathbf{k})-(2\mathbf{i}-4\mathbf{j}+4\mathbf{k}))=\pm(-\mathbf{i}+6\mathbf{j}-9\mathbf{k})$	M1 A1
	$d=AB\cos\theta=\dfrac{\overrightarrow{AB}.\mathbf{n}}{\lvert\mathbf{n}\rvert}=\dfrac{\pm(-\mathbf{i}+6\mathbf{j}-9\mathbf{k}).(3\mathbf{i}-15\mathbf{j}-18\mathbf{k})}{\sqrt{3^2+15^2+18^2}}$ oe	M1
	$=\dfrac{\pm(-3-90+162)}{\sqrt{558}}=\dfrac{\pm 69}{\sqrt{558}}=...$	M1
	$=\dfrac{23}{\sqrt{62}}$ or $\dfrac{23\sqrt{62}}{62}$	A1
		(5)

Way 3	$(2\mathbf{i}-4\mathbf{j}+4\mathbf{k})+\mu(8\mathbf{i}-2\mathbf{j}+3\mathbf{k})-((\mathbf{i}+2\mathbf{j}-5\mathbf{k})+\lambda(5\mathbf{i}+\mathbf{j}))$ $=(1+8\mu-5\lambda)\mathbf{i}+(-6-2\mu-\lambda)\mathbf{j}+(9+3\mu)\mathbf{k}$	**M1 A1**
	$((1+8\mu-5\lambda)\mathbf{i}+(-6-2\mu-\lambda)\mathbf{j}+(9+3\mu)\mathbf{k}).(5\mathbf{i}+\mathbf{j})=0$ $\Rightarrow 38\mu-26\lambda=1$ $((1+8\mu-5\lambda)\mathbf{i}+(-6-2\mu-\lambda)\mathbf{j}+(9+3\mu)\mathbf{k}).(8\mathbf{i}-2\mathbf{j}+3\mathbf{k})=0$ $\Rightarrow 77\mu-38\lambda=-47$ $\Rightarrow \lambda=-\frac{207}{62},\ \mu=-\frac{70}{31}$	**M1**
	$(2\mathbf{i}-4\mathbf{j}+4\mathbf{k})+\mu(8\mathbf{i}-2\mathbf{j}+3\mathbf{k})-((\mathbf{i}+2\mathbf{j}-5\mathbf{k})+\lambda(5\mathbf{i}+\mathbf{j}))$ $=-\frac{23}{62}\mathbf{i}+\frac{115}{62}\mathbf{j}+\frac{69}{31}\mathbf{k}$ $d=\sqrt{\left(\frac{23}{62}\right)^2+\left(\frac{115}{62}\right)^2+\left(\frac{69}{31}\right)^2}$	**M1**
	$=\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$	**A1**
		(5)
		(10 marks)

Notes:

Accept equivalent vector notation, e.g. column vectors, throughout.

(a)

M1: Any correct method to find a vector perpendicular to the two direction vectors of the lines. Look for the cross product between the two direction vectors, but may use dot products and solving equations. In the latter case the method should lead to values for u, v and w.

For the vector product, if no method is shown look for at least 2 correct components.

A1: Any correct vector, a scalar multiple of $-\mathbf{i}+5\mathbf{j}+6\mathbf{k}$

(b)

B1: Any correct equation. Must have $\mathbf{r}=\ldots$ or e.g. $\begin{pmatrix}x\\y\\z\end{pmatrix}=\ldots$

M1: Uses their normal vector from (a) with any point on the plane (probably $(2\mathbf{i}-4\mathbf{j}+4\mathbf{k})$ to find p

Condone slips with the calculation so $(2\mathbf{i}-4\mathbf{j}+4\mathbf{k}).(3\mathbf{i}-15\mathbf{j}-18\mathbf{k})$ evaluated as a scalar is sufficient for M1. May also be implied by $p=-6$

A1: Any correct equation of the correct form.

(c)

Way 1

M1: Uses the plane equation from (b) (or otherwise) OR the parallel plane containing l_1 to find the distance of one of these planes to the origin.

A1: Correct distance between one of the planes and the origin, accept ± here.

M1: Attempts distance of both the parallel planes containing l_1 and l_2 from the origin.

M1: Correct method for finding the distance between lines – i.e. subtracts their distances either way round.

A1: Correct answer. Accept $\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$

Way 2

M1: Subtracts position vectors of points on the lines (either way around). Implied by two correct coordinates if method not shown. (Forms suitable hypotenuse.)

A1: Correct vector or as coordinates, either direction.

M1: Correct formula for the distance using their vectors, $d = AB\cos\theta = \frac{\overrightarrow{AB}.\mathbf{n}}{|\mathbf{n}|}$ with their $\overrightarrow{AB}$ and **n**.

M1: Complete evaluation of the formula.

A1: Correct answer. Accept $\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$ but must be positive.

Way 3

M1: Subtracts position vectors of general points on each line (either way around). Implied by two correct coordinates if method not shown.

A1: Correct vector or as coordinates, either direction.

M1: Forms scalar product of the general vector with both direction vectors, sets = 0 and solves simultaneously

M1: Substitutes the values of their parameters back into the general vector and attempts its magnitude

A1: Correct answer. Accept $\frac{23}{\sqrt{62}}$ or $\frac{23\sqrt{62}}{62}$ but must be positive.

Question	Scheme	Marks
6(a) Way 1	$I_n=\int_0^{\sqrt{\frac{\pi}{2}}} x^{n-1}.x\cos\left(x^2\right)\,\mathrm{d}x=\left[x^{n-1}.\frac{1}{2}\sin\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}-\int_0^{\sqrt{\frac{\pi}{2}}}(n-1)x^{n-2}.\frac{1}{2}\sin\left(x^2\right)\,\mathrm{d}x$	M1A1
	$=\left[x^{n-1}.\frac{1}{2}\sin\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}-\frac{1}{2}(n-1)\int_0^{\sqrt{\frac{\pi}{2}}} x^{n-3}.x\sin\left(x^2\right)\,\mathrm{d}x$ $=\left[x^{n-1}.\frac{1}{2}\sin\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}-\frac{1}{2}(n-1)\left(\underline{\left[x^{n-3}.-\frac{1}{2}\cos\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}-\int_0^{\sqrt{\frac{\pi}{2}}}(n-3)x^{n-4}.-\frac{1}{2}\cos\left(x^2\right)\,\mathrm{d}x}\right)$	<u>dM1A1</u>
	$=\left(\frac{1}{2}\left(\sqrt{\frac{\pi}{2}}\right)^{n-1}\sin\frac{\pi}{2}-0\right)-\frac{1}{2}(n-1)\left[(0-0)+\frac{1}{2}(n-3)I_{n-4}\right]$	dM1
	$=\frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{n-1}{2}}-\frac{1}{4}(n-1)(n-3)I_{n-4}$ *	A1*
		(6)
Way 2	$I_n=\left[\frac{x^{n+1}}{n+1}.\cos\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}-\int_0^{\sqrt{\frac{\pi}{2}}}\frac{x^{n+1}}{n+1}.-2x\sin\left(x^2\right)\,\mathrm{d}x$	M1A1
	$=\left[\frac{x^{n+1}}{n+1}.\cos\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}+\frac{2}{n+1}\int_0^{\sqrt{\frac{\pi}{2}}} x^{n+2}\sin\left(x^2\right)\,\mathrm{d}x$ $=\left[\frac{x^{n+1}}{n+1}.\cos\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}+\frac{2}{n+1}\left(\underline{\left[\frac{x^{n+3}}{n+3}.\sin\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}-\int_0^{\sqrt{\frac{\pi}{2}}}\frac{x^{n+3}}{n+3}.2x\cos\left(x^2\right)\,\mathrm{d}x}\right)$	<u>dM1A1</u>
	$=(0-0)+\frac{2}{n+1}\left(\frac{1}{n+3}\left(\sqrt{\frac{\pi}{2}}\right)^{n+3}\sin\frac{\pi}{2}-0-\frac{2}{n+3}I_{n+4}\right)$	dM1
	$\Rightarrow I_{n+4}=\frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{n+3}{2}}-\frac{1}{4}(n+1)(n+3)I_n$ so replacing n by $n-4$ gives $I_n=\frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{n-1}{2}}-\frac{1}{4}(n-1)(n-3)I_{n-4}$ *	A1*
		(6)
(b)	$I_1=\int_0^{\sqrt{\frac{\pi}{2}}} x\cos\left(x^2\right)\,\mathrm{d}x=\left[\frac{1}{2}\sin\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}=\frac{1}{2}$	B1
	$I_5=\frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{5-1}{2}}-\frac{1}{4}(5-1)(5-3)\times\text{"}\frac{1}{2}\text{"}$	M1
	$=\frac{\pi^2}{8}-1$ oe e.g. $\frac{\pi^2-8}{8}$, $\frac{1}{2}\left(\frac{\pi}{2}\right)^2-1$	A1
		(3)

(9 marks)

Notes:

(a) Way 1

M1: Applies integration by parts in the correct direction having made the 'split' and obtains:

$$\left[\pm\alpha x^{n-1}\sin\left(x^2\right)\right] \pm\beta\int x^{n-2}.\sin\left(x^2\right)\,dx$$

A1: Fully correct expression

dM1: Applies integration by parts in the correct direction to $\beta\int x^{n-2}.\sin\left(x^2\right)\,dx$ and obtains:

$$\left[\pm\alpha x^{n-3}\cos\left(x^2\right)\right] \pm\beta\int x^{n-4}\cos\left(x^2\right)\,dx$$

Depends on the previous M mark.

A1: Correct second application of parts e.g.

$$\int x^{n-2}.\sin\left(x^2\right)\,dx = \left[x^{n-3}.-\frac{1}{2}\cos\left(x^2\right)\right] - \int (n-3)x^{n-4}.-\frac{1}{2}\cos\left(x^2\right)\,dx$$

dM1: Applies the limits completely to their result and replaces final integral by I_{n-4}. The substitution of limits may have been carried out in stages throughout the work, or may be applied after integration by parts twice has been carried out. **Depends on both previous M marks. There must some explicit evidence that the limits have been applied but this may be taken from either the** $\left[x^{n-1}.\frac{1}{2}\sin\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}$ = e.g. $\sqrt{\frac{\pi}{2}}^{\,n-1}.\frac{1}{2}\sin\left(\sqrt{\frac{\pi}{2}}^{\,2}\right)$, $\sqrt{\frac{\pi}{2}}^{\,n-1}.\frac{1}{2}$, $\frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} - 0$

or $\left[x^{n-3}.-\frac{1}{2}\cos\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}$ = e.g. $0-0,\ 0$

A1*: Achieves the printed answer from completely correct work with no errors seen and evidence of the given limits being applied.

Way 2

M1: Applies integration by parts in the correct direction and obtains:

$$\left[\pm\alpha x^{n+1}\cos\left(x^2\right)\right] \pm\beta\int x^{n+1}.x\sin\left(x^2\right)\,dx$$

A1: Fully correct expression

dM1: Applies integration by parts in the correct direction to $\beta\int x^{n+1}.x\sin\left(x^2\right)\,dx$ and obtains:

$$\left[\pm\alpha x^{n+3}\sin\left(x^2\right)\right] \pm\beta\int x^{n+3}.x\cos\left(x^2\right)\,dx$$

Depends on the previous M mark.

A1: Correct second application of parts e.g.

$$\int x^{n+2}.\sin\left(x^2\right)\,dx = \left[\frac{x^{n+3}}{n+3}.\sin\left(x^2\right)\right] - \int \frac{x^{n+3}}{n+3}.2x\cos\left(x^2\right)\,dx$$

dM1: Applies the limits completely to their result and replaces final integral by I_{n+4}. The substitution of limits may have been carried out in stages throughout the work, or may be applied after integration by parts twice has been carried out. **Depends on both previous M marks.**
There must some explicit evidence that the limits have been applied but this may be taken from either the $\left[\frac{x^{n+1}}{n+1}.\cos\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}}$ = e.g. $0-0, 0$ or

$$\left[\frac{x^{n+3}}{n+3}.\sin\left(x^2\right)\right]_0^{\sqrt{\frac{\pi}{2}}} = \text{e.g.} \frac{\sqrt{\frac{\pi}{2}}^{n+3}}{n+3}.\sin\left(\sqrt{\frac{\pi}{2}}^2\right), \frac{\sqrt{\frac{\pi}{2}}^{n+3}}{n+3}.\sin\left(\frac{\pi}{2}\right), \frac{\sqrt{\frac{\pi}{2}}^{n+3}}{n+3}.(1), \frac{\left(\frac{\pi}{2}\right)^{\frac{n+3}{2}}}{n+3}-0$$

A1*: Achieves the printed answer from completely correct work with no errors and evidence of the given limits being applied with a clear statement that n is replaced by $n-4$
(b)
B1: Correct I_1. May be seen after attempting the reduction.
M1: Applies the reduction formula with their I_1 and $n = 5$ to reach a value. Condone slips with evaluating $\frac{1}{4}(n-1)(n-3)$ as long as the intention is clear.
A1: Correct answer.

Note: Beware incorrect work in (a) leading to what appears to be a correct form e.g.

$$I_n = \int_0^{\sqrt{\frac{\pi}{2}}} x^n \cos\left(x^2\right)\,\mathrm{d}x = \left[x^n.\frac{\sin\left(x^2\right)}{2x}\right]_0^{\sqrt{\frac{\pi}{2}}} - \int_0^{\sqrt{\frac{\pi}{2}}} nx^{n-1}.\frac{\sin\left(x^2\right)}{2x}\,\mathrm{d}x$$

This scores M0 at the start and hence will usually score no marks in part (a)

Question	Scheme	Marks
7(a)	$b^2 = a^2\left(e^2 - 1\right) \Rightarrow e^2 = \frac{25}{a^2} + 1 = \frac{25 + a^2}{a^2}$ oe	B1
		(1)
(b)	$x = (\pm)\frac{a}{e}$ $\quad\quad$ $\frac{x}{a} = (\pm)\frac{y}{5}$	B1
	$\frac{a}{e} \times \frac{1}{a} = \pm\frac{y}{5} \Rightarrow y = \pm\frac{5}{e} \Rightarrow AA? = \quad \times\frac{5}{e}$ or $\frac{5}{e} - \left(-\frac{5}{e}\right)$	M1
	$= \frac{10}{e}$	A1
		(3)
(c)	$\frac{1}{2} \times \frac{"10"}{e} \times \left(ae + \frac{a}{e}\right)$ or e.g. $\frac{1}{2} \times \frac{10a}{\sqrt{25 + a^2}} \times \left(\sqrt{25 + a^2} + \frac{a^2}{\sqrt{25 + a^2}}\right)$	M1
	$\frac{1}{2}\frac{10}{e}\left(ae + \frac{a}{e}\right) = \frac{164}{3} \Rightarrow 15\left(a + \frac{a}{e^2}\right) = 164$ or $\frac{1}{2} \times \frac{10a}{\sqrt{25 + a^2}} \times \left(\sqrt{25 + a^2} + \frac{a^2}{\sqrt{25 + a^2}}\right) = \frac{164}{3}$	M1
	$\Rightarrow 15a\left(1 + \frac{a^2}{25 + a^2}\right) = 164$	A1 (M1 on EPEN)
	$\Rightarrow 15a\left(\frac{25 + 2a^2}{25 + a^2}\right) = 164 \Rightarrow 375a + 30a^3 = 164\left(25 + a^2\right)$ $\Rightarrow 30a^3 - 164a^2 + 375a - 4100 = 0$ *	A1*
		(4)
(d)	$30a^3 - 164a^2 + 375a - 4100 = (3a - 20)\left(10a^2 + 12a + 205\right)$	B1 (M1 on EPEN)
	$12^2 - 4(10)(205) = ...$ $10a^2 + 12a + 205 = 10\left(\left(a + \frac{12}{20}\right)^2 - \frac{144}{400}\right) + 205$	M1
	E.g. $12^2 - 4(10)(205) < 0$ so there are no other roots of the equation. Hence $a = \frac{20}{3}$ is only possible value.	A1
		(3)
		(11 marks)

Notes:

(a)
B1: Correct expression.

(b)

B1: Identifies at least one correct equation for a directrix and at least one asymptote, stated or used – including the $b = 5$.

M1: Solves to find y coordinates of A and A' or just one of these and doubles to get length. Allow if b is used rather than 5.

A1: Correct length (from subtracting or doubling). Must be positive.

(c)

M1: Uses focus $(-ae, 0)$ and directrix $x = \frac{a}{e}$ (allow if the alternative pair is used) with their length from (b), to form a **correct or correct ft** expression for the area of triangle AFA'.

M1: Sets their area equation equal to $\frac{164}{3}$ to obtain an equation in e^2 and a.

Their attempt at the area must be of the form $\frac{1}{2} \times \frac{"10"}{e} \times \pm\left(ae \pm \frac{a}{e}\right)$

Alternatively, allow an equation in just a^2 if $e = \sqrt{\frac{25+a^2}{a^2}}$ is substituted first.

A1(M1 on EPEN): Correct equation in terms of a only. Allow any correct form.

A1*: Correct result achieved with no errors seen and sufficient working shown.

(d)

B1(M1 on EPEN): A correct method for showing that $a = \frac{20}{3}$ is a solution of the equation.

Examples:

$$30a^3 - 164a^2 + 375a - 4100 = (3a - 20)(10a^2 + 12a + 205)$$

$$30a^3 - 164a^2 + 375a - 4100 = \left(a - \frac{20}{3}\right)(30a^2 + 36a + 615)$$

$$\mathrm{f}\left(\frac{20}{3}\right) = \frac{80000}{9} - \frac{65600}{9} + 2500 - 4100 = 0$$

Or e.g. long division and obtains correct quotient and no remainder

M1: A correct method for showing there are no other roots. May use completing the square (as in scheme) or attempt discriminant or differentiation,

e.g. $\frac{\mathrm{d}}{\mathrm{d}a}(eqn) = 90a^2 - 328a + 375 = 90\left(a - \frac{82}{45}\right)^2 + \frac{3427}{45} > 0$ so strictly increasing hence only one solution.

If using discriminant then values must be used i.e. not just $b^2 - 4ac < 0$

An attempt at the discriminant may be seen as part of the quadratic formula e.g.

$$a = \frac{-12 \pm \sqrt{12^2 - 4(10)(205)}}{2(10)}$$

A1: All work correct with **reason** and **conclusion** made that $a = \frac{20}{3}$ is the only possible value. If the discriminant is evaluated then it must be correct. For reference $12^2 - 4(10)(205) = -8056$ and $36^2 - 4(30)(615) = -72504$ but note that e.g. $12^2 - 4(10)(205) < 0$ with a conclusion is acceptable.

Note that just using a calculator to solve the cubic generally scores no marks.

Question	Scheme	Marks
8(a)	$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1-k\sqrt{x}^2}} \times ...x^{-\frac{1}{2}}$ or $\cos y = 2x^{\frac{1}{2}} \Rightarrow \pm\sin y\frac{dy}{dx} = ..x^{-\frac{1}{2}}$	M1
	$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1-4x}} \times \left(Kx^{-\frac{1}{2}}\right)$ or $\frac{dy}{dx} = \pm\frac{Kx^{-\frac{1}{2}}}{\sqrt{1-\left(2\sqrt{x}\right)^2}}$	dM1
	$\frac{dy}{dx} = -\frac{1}{\sqrt{x}\sqrt{1-4x}}$ oe e.g. $\frac{dy}{dx} = -\frac{1}{\sqrt{x-4x^2}}$	A1
		(3)
(b) Way 1	$\int y\,dx = \int 1\times\arccos\left(2\sqrt{x}\right)dx = x\arccos\left(2\sqrt{x}\right) - \int x\frac{-1}{\sqrt{x}\sqrt{1-4x}}dx$	M1
	$= x\arccos\left(2\sqrt{x}\right) + \int\frac{\sqrt{x}}{\sqrt{1-4x}}dx$ *	A1*
		(2)
Way 2	$\frac{d}{dx}\left(x\arccos\left(2\sqrt{x}\right)\right) = 1.\arccos\left(2\sqrt{x}\right) + x.\frac{-1}{\sqrt{x}\sqrt{1-4x}}$	M1
	$\Rightarrow \int\arccos\left(2\sqrt{x}\right)dx = x\arccos\left(2\sqrt{x}\right) + \int\frac{\sqrt{x}}{\sqrt{1-4x}}dx$ *	A1*
		(2)
(c)	$\frac{1}{2\sqrt{x}}\frac{dx}{d\theta} = -\frac{1}{2}\sin\theta,\ dx = -\sqrt{x}\sin\theta\,d\theta,\ \frac{dx}{d\theta} = -\frac{1}{2}\sin\theta\cos\theta$ $\frac{dx}{d\theta} = -\frac{1}{4}\sin 2\theta$	B1
	$\int\frac{\sqrt{x}}{\sqrt{1-4x}}dx = \int\frac{-\left(\frac{1}{2}\cos\theta\right)^2\sin\theta}{\sqrt{1-4\left(\frac{1}{2}\cos\theta\right)^2}}d\theta$	M1
	$= -\frac{1}{4}\int\frac{\cos^2\theta\sin\theta}{\sqrt{1-\cos^2\theta}}d\theta = -\frac{1}{4}\int\cos^2\theta\,d\theta$	A1
	$x = 0 \Rightarrow \theta = \frac{\pi}{2}$ $x = \frac{1}{8} \Rightarrow \theta = \frac{\pi}{4}$ So $\int_0^{\frac{1}{8}}\frac{\sqrt{x}}{\sqrt{1-4x}}dx = \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$	A1
		(4)

(d)	$\frac{1}{4}\int\frac{1}{2}(1+\cos 2\theta)\,\mathrm{d}\theta = K\left(\theta \pm \frac{1}{2}\sin 2\theta\right)$	**M1**
	$\int_0^{\frac{1}{8}}\frac{\sqrt{x}}{\sqrt{1-4x}}\,\mathrm{d}x = \frac{1}{8}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \ldots\left(=\frac{\pi}{32}-\frac{1}{16}\right)$ or e.g. $\int_0^{\frac{1}{8}}\frac{\sqrt{x}}{\sqrt{1-4x}}\,\mathrm{d}x = \frac{1}{8}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{1}{8}\left[\arccos 2\sqrt{x}+\frac{1}{2}\sin 2\arccos 2\sqrt{x}\right]_0^{\frac{1}{8}}$ $=\ldots\left(=-\frac{1}{8}\left(\frac{\pi}{4}+\frac{1}{2}-\frac{\pi}{2}\right)\right)$	**dM1**
	$\Rightarrow \int_0^{\frac{1}{8}}\arccos\left(2\sqrt{x}\right)\mathrm{d}x = \left[x\arccos 2\sqrt{x}\right]_0^{\frac{1}{8}}+\frac{\pi}{32}-\frac{1}{16}=\frac{1}{8}\arccos\frac{1}{\sqrt{2}}-0+\frac{\pi}{32}-\frac{1}{16}$	**dM1**
	$=\frac{\pi}{16}-\frac{1}{16}$ oe	**A1**
		(4)
		(13 marks)

Notes:

(a)

M1: Attempts to apply the arccos derivative formula together with chain rule. Look for $\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\frac{1}{\sqrt{1-k\sqrt{x}^2}}\times \mathrm{f}(x)$ where f(*x*) is an attempt at differentiating $2\sqrt{x}$ where $\mathrm{f}(x)\neq\alpha\sqrt{x}$

Note that *k* may be 1 for this mark.

Alternatively, takes cosine of both sides and differentiates to the form shown in the scheme.

dM1: Correct form for the overall derivative achieved, may be errors in sign or constants with $k\neq 1$

Alternatively, divides through by sin *y* and applies Pythagorean identity to achieve derivative in terms of *x*.

A1: Correct derivative, but need not be simplified. Award when first seen and isw.

(b) Way 1

M1: Attempts to apply integration by parts to $1\times\arccos\left(2\sqrt{x}\right)$.

Look for $x\arccos\left(2\sqrt{x}\right)-\int x\,\text{"their (a)"}\,\mathrm{d}x$ or $u=\arccos\left(2\sqrt{x}\right)\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\text{part}(a),\ \frac{\mathrm{d}v}{\mathrm{d}x}=1\Rightarrow v=x$

A1*: Correct work leading to the printed answer. There must be a clear statement for the integration by parts before the given answer is stated.

So e.g. $u=\arccos\left(2\sqrt{x}\right)\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\text{part}(a),\ \frac{\mathrm{d}v}{\mathrm{d}x}=1\Rightarrow v=x$

$\Rightarrow\int\arccos\left(2\sqrt{x}\right)\mathrm{d}x=x\arccos\left(2\sqrt{x}\right)+\int\frac{\sqrt{x}}{\sqrt{1-4x}}\,\mathrm{d}x^*$ scores M1A0

You can condone $\int\arccos\left(2\sqrt{x}\right)\mathrm{d}x=x\arccos\left(2\sqrt{x}\right)+\int\frac{x^{\frac{1}{2}}}{\sqrt{1-4x}}\,\mathrm{d}x^*$

Way 2

M1: Applies the product rule to $x\arccos\left(2\sqrt{x}\right)$, look for $1.\arccos\left(2\sqrt{x}\right)+x.$"their (a)".

A1*: Rearranges and integrates to achieve the given result, with no errors seen.

(c)

B1: Any correct expression involving $\mathrm{d}x$ and $\mathrm{d}\theta$, see examples in scheme.

M1: Makes a complete substitution in the integral $\int \frac{\sqrt{x}}{\sqrt{1-4x}}\,\mathrm{d}x$ to achieve an integral in θ only.

Ignore attempts at substitution into the $x\arccos\left(2\sqrt{x}\right)$.

A1: A correct simplified integral aside from limits. May be implied by e.g. $\frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta$

Note that this mark depends on the B mark.

A1: Finds correct limits for θ and applies to the integral by reversing the sign – i.e. correct answer with limits and sign all correct. Accept equivalent limits e.g. $-\frac{\pi}{4}$ to $-\frac{\pi}{4}$ or $\frac{\pi}{2}$ to $\frac{3\pi}{4}$

Note that this mark depends on the B mark.

(d)

M1: Applies double angle identity to get the integral in a suitable form and attempts to integrate. Accept $\cos^2\theta=\frac{1}{2}(\pm1\pm\cos 2\theta)$ used as identity and look for $1\to\theta$ and $\cos 2\theta\to\pm\frac{1}{2}\sin 2\theta$

dM1: Applies their limits (either way round) to their integral in θ or reverse substitution and applies limits 0 and $\frac{1}{8}$.

Depends on the previous method mark.

dM1: Applies limits of 0 and $\frac{1}{8}$ to the $x\arccos\left(2\sqrt{x}\right)$ to obtain a value (or their limits either way round if they applied the substitution to this to obtain a value) and combines with the result of the other integral.

Depends on both previous method marks.

A1: Correct final answer.

Question Number	Scheme	Notes	Marks
1(a)	$8\cosh^4 x = 8\left(\frac{e^x + e^{-x}}{2}\right)^4 = \frac{8}{16}\left(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}\right)$ Applies $\cosh x = \frac{e^x + e^{-x}}{2}$ and attempts to expand the bracket to at least 4 different and no more than 5 different terms of the correct form but they may be "uncollected" depending on how they do the expansion. Allow unsimplified terms e.g. $\left(e^x\right)^3 e^{-x}$. May see $8\left(\frac{e^x + e^{-x}}{2}\right)^2\left(\frac{e^x + e^{-x}}{2}\right)^2$ but must attempt to expand as above		M1
	$= \frac{1}{2}\left(e^{4x} + e^{-4x}\right) + 4\left(\frac{e^{2x} + e^{-2x}}{2}\right) + 3 = \ldots$	Collects appropriate terms and reaches the form $\cosh 4x + p\cosh 2x + q$ or obtains values of p and q.	M1
	$= \cosh 4x + 4\cosh 2x + 3$	Correct expression or values e.g. $p = 4$ and $q = 3$	A1
	No marks are available in (a) if exponentials are not used but note that they may appear in combination with the use of hyperbolic identities e.g.: $8\cosh^4 x = 8\left(\cosh^2 x\right)^2 = 8\left(\frac{\cosh 2x + 1}{2}\right)^2 = 2\left(\frac{e^{2x} + e^{-2x}}{2} + 1\right)^2$ $= 2\left(\frac{e^{4x} + 2 + e^{-4x}}{4} + e^{2x} + e^{-2x} + 1\right) = \frac{e^{4x} + e^{-4x}}{2} + 4\left(\frac{e^{2x} + e^{-2x}}{2}\right) + 2$ $= \cosh 4x + 4\cosh 2x + 3$ **Allow to "meet in the middle" e.g. expands as above and compares with** $\frac{1}{2}\left(e^{4x} + e^{-4x}\right) + p\left(\frac{e^{2x} + e^{-2x}}{2}\right) + q \Rightarrow p = \ldots, q = \ldots$ **but to score any marks the expansion must be attempted.**		
			(3)

(b) Way 1	$\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow 8\cosh^4 x - 4\cosh 2x - 3 - 17\cosh 2x + 9 = 0$ $\Rightarrow 8\cosh^4 x - 21\cosh 2x + 6 = 0 \Rightarrow 8\cosh^4 x - 21\left(2\cosh^2 x - 1\right) + 6 = 0$ Uses **their** result from part (a) and $\cosh 2x = \pm 2\cosh^2 x \pm 1$ to obtain a quadratic equation in $\cosh^2 x$ **or** $\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow 2\left(2\cosh^2 x - 1\right)^2 - 1 - 17\left(2\cosh^2 x - 1\right) + 9 = 0$ Uses $\cosh 4x = \pm 2\cosh^2 2x \pm 1$ and $\cosh 2x = \pm 2\cosh^2 x \pm 1$ to obtain a quadratic equation in $\cosh^2 x$		M1
	$\Rightarrow 8\cosh^4 x - 42\cosh^2 x + 27 = 0$	Correct 3TQ in $\cosh^2 x$	A1
	$\Rightarrow 8\cosh^4 x - 42\cosh^2 x + 27 = 0$ $\Rightarrow \cosh^2 x = \frac{9}{2}\left(, \frac{3}{4}\right)$	Solves 3TQ in $\cosh^2 x$ (apply usual rules if necessary) to obtain $\cosh^2 x = k \ \left(k \in \mathbb{R} \text{ and } > 1\right)$. May be implied by their values – check if necessary.	M1
	$\cosh^2 x = \frac{9}{2} \Rightarrow \cosh x = \frac{3}{\sqrt{2}} \Rightarrow x = \pm\ln\left(\frac{3}{\sqrt{2}} + \sqrt{\frac{9}{2} - 1}\right)$ or $\cosh x = \frac{3}{\sqrt{2}} \Rightarrow \frac{e^x + e^{-x}}{2} = \frac{3}{\sqrt{2}} \Rightarrow \sqrt{2}e^{2x} - 6e^x + \sqrt{2} = 0 \Rightarrow e^x = \ldots \Rightarrow x = \ldots$ or $\cosh^2 x = \frac{9}{2} \Rightarrow \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{9}{2} \Rightarrow e^{4x} - 16e^{2x} + 1 = 0 \Rightarrow e^{2x} = \ldots \Rightarrow x = \ldots$ Takes square root to obtain $\cosh x = k \ \left(k > 1\right)$ and applies the correct logarithmic form for arcosh or uses the correct exponential form for cosh x to obtain at least one value for x The root(s) must be real to score this mark.		M1
	$x = \pm\ln\left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{14}}{2}\right)$ Both correct and exact including brackets. Accept simplified equivalents e.g. $x = \ln\left(\frac{3}{\sqrt{2}} \pm \frac{\sqrt{7}}{\sqrt{2}}\right)$ but withhold this mark if additional answers are given unless they are the same e.g. allow $x = \pm\ln\left(\frac{3\sqrt{2}}{2} \pm \frac{\sqrt{14}}{2}\right)$		A1
			(5)

(b) Way 2	$\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow 2\cosh^2 2x - 1 - 17\cosh 2x + 9 = 0$ Applies $\cosh 4x = \pm 2\cosh^2 2x \pm 1$ to obtain a quadratic equation in $\cosh 2x$		M1
	$2\cosh^2 2x - 17\cosh 2x + 8 = 0$	Correct 3TQ in $\cosh 2x$	A1
	$2\cosh^2 2x - 17\cosh 2x + 8 = 0$ $\Rightarrow \cosh 2x = 8\left(, \frac{1}{2}\right)$	Solves 3TQ in $\cosh 2x$ (apply usual rules if necessary) to obtain $\cosh 2x = k \quad (k \in \mathbb{R}$ and $> 1)$	M1
	$\cosh 2x = 8 \Rightarrow 2x = \pm\ln\left(8 + \sqrt{8^2 - 1}\right)$ or $\cosh 2x = 8 \Rightarrow \frac{e^{2x} + e^{-2x}}{2} = 8 \Rightarrow e^{4x} - 16e^{2x} + 1 = 0 \Rightarrow e^{2x} = \ldots \Rightarrow 2x = \ldots$ Applies the correct logarithmic form for arcosh from $\cosh 2x = k \quad (k > 1)$ or uses the correct exponential form for $\cosh 2x$ to obtain at least one value for $2x$ The root(s) must be real to score this mark.		M1
	$x = \pm\frac{1}{2}\ln\left(8 + 3\sqrt{7}\right)$ or e.g. $x = \pm\ln\left(8 + 3\sqrt{7}\right)^{\frac{1}{2}}$	Both correct and exact with brackets. Accept simplified equivalents e.g. $x = \frac{1}{2}\ln\left(8 \pm \sqrt{63}\right)$ but withhold this mark if additional answers are given unless they are the same as above.	A1
(b) Way 3	$\cosh 4x - 17\cosh 2x + 9 = 0 \Rightarrow \frac{e^{4x} + e^{-4x}}{2} - \frac{17}{2}\left(e^{2x} + e^{-2x}\right) + 9 = 0$ $\Rightarrow e^{8x} - 17e^{6x} + 18e^{4x} - 17e^{2x} + 1 = 0$ M1: Applies the correct exponential forms and attempts a quartic equation in e^{2x} A1: Correct equation		M1A1
	$e^{8x} - 17e^{6x} + 18e^{4x} - 17e^{2x} + 1 = 0$ $\Rightarrow e^{2x} = 8 \pm 3\sqrt{7}, \ldots$	Solves and proceeds to a value for e^{2x} where $e^{2x} > 1$ and real.	M1
	$\Rightarrow e^{2x} = 8 \pm 3\sqrt{7} \Rightarrow 2x = \ln\left(8 \pm 3\sqrt{7}\right)$	Takes ln's to obtain at least one value for $2x$ The root(s) must be real to score this mark.	M1
	$x = \frac{1}{2}\ln\left(8 \pm 3\sqrt{7}\right)$ or e.g. $x = \ln\left(8 \pm 3\sqrt{7}\right)^{\frac{1}{2}}$	Both correct and exact with brackets. Accept simplified equivalents e.g. $x = \pm\frac{1}{2}\ln\left(8 + 3\sqrt{7}\right)$ but withhold this mark if additional answers are given unless they are the same as above.	A1
			Total 8

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td>2</td><td colspan="2">$\frac{dx}{d\theta}=\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}-\cos\theta$
Correct derivative.
Do not condone missing brackets e.g. $\frac{dx}{d\theta}=\frac{1}{\sec\theta+\tan\theta}\times\sec\theta\tan\theta+\sec^2\theta-\cos\theta$
unless a correct expression is implied by subsequent work. Award when a correct expression is seen but note that other forms are possible
e.g. $\sec\theta-\cos\theta,\ \tan\theta\sin\theta$</td><td>B1</td></tr>
<tr><td></td><td colspan="2">$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\left(\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}-\cos\theta\right)^2+(-\sin\theta)^2$
Attempts $\frac{dy}{d\theta}$ and then $\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2$</td><td>M1</td></tr>
<tr><td></td><td colspan="2">$S=(2\pi)\int\cos\theta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$
$=(2\pi)\int\cos\theta\sqrt{\left(\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}-\cos\theta\right)^2+(-\sin\theta)^2}\,d\theta$
Applies a correct surface area formula using their $\frac{dx}{d\theta}$ and their $\frac{dy}{d\theta}$ with or without the 2π
For reference: $\sqrt{\left(\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}-\cos\theta\right)^2+(-\sin\theta)^2}=\tan\theta$
Allow π in front of the integral but must be an integral</td><td>M1</td></tr>
<tr><td></td><td>$(2\pi)\int\sin\theta\ d\theta$</td><td>Fully correct simplified integral with or without the 2π</td><td>A1</td></tr>
<tr><td></td><td>$=(2\pi)[-\cos\theta](+c)$</td><td>Correct integration with or without the 2π</td><td>A1</td></tr>
<tr><td></td><td colspan="2">$(2\pi)[-\cos\theta]_0^{\frac{\pi}{4}}=(2\pi)\left(-\frac{1}{\sqrt{2}}+1\right)$ Applies the limits 0 and $\frac{\pi}{4}$.
Must see evidence of both limits if necessary but condone e.g. $(2\pi)\left(-\frac{1}{\sqrt{2}}-1\right)$
Depends on both previous method marks.</td><td>dM1</td></tr>
<tr><td></td><td>TSA =
$2\pi\left(-\frac{1}{\sqrt{2}}+1\right)+\pi\times1^2+\pi\times\left(\frac{1}{\sqrt{2}}\right)^2$</td><td>Correct expressions for the 2 “ends” and adds these to their curved surface area. Depends on the previous method mark.</td><td>dM1</td></tr>
<tr><td></td><td>$=\frac{\pi}{2}\left(7-2\sqrt{2}\right)$</td><td>Correct answer in the required form or correct values for p and q.</td><td>A1</td></tr>
<tr><td></td><td colspan="2"><u>Note:</u>
The final answer should follow correct work. The final mark should be withheld following e.g. $\frac{dy}{d\theta}$ clearly seen as +sin θ or $\int\sin\theta\ d\theta=+\cos\theta$
<u>Note:</u>
Without the “ends” the answer is $\frac{\pi}{2}\left(4-2\sqrt{2}\right)$ (usually scores 6/8)</td><td></td></tr>
<tr><td></td><td></td><td></td><td>(8)</td></tr>
<tr><td></td><td></td><td></td><td>Total 8</td></tr>
</table>

Alternative for first 4 marks:

	$\frac{dx}{d\theta}=\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}-\cos\theta$ Correct derivative. Do not condone missing brackets e.g. $\frac{dx}{d\theta}=\frac{1}{\sec\theta+\tan\theta}\times\sec\theta\tan\theta+\sec^2\theta-\cos\theta$ unless a correct expression is implied by subsequent work. Award when a correct expression is seen but note that other forms are possible e.g. $\sec\theta-\cos\theta,\ \tan\theta\sin\theta$		B1
	$1+\left(\frac{dy}{dx}\right)^2=1+\left(\frac{-\sin\theta}{\sec\theta-\cos\theta}\right)^2$ Attempts $1+\left(\frac{dy}{dx}\right)^2$ with $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$		M1
	$S=(2\pi)\int\cos\theta\sqrt{1+\left(\frac{dy}{dx}\right)^2}\frac{dx}{d\theta}d\theta$ $=(2\pi)\int\cos\theta\sqrt{1+\left(\frac{-\sin\theta}{\sec\theta-\cos\theta}\right)^2}(\sec\theta-\cos\theta)d\theta$ Applies a correct surface area formula using their $\frac{dx}{d\theta}$ and their $\frac{dy}{dx}$ with or without the 2π For reference: $\sqrt{1+\left(\frac{-\sin\theta}{\sec\theta-\cos\theta}\right)^2}(\sec\theta-\cos\theta)=\tan\theta$ Allow π in front of the integral but must be an integral		M1
	$(2\pi)\int\sin\theta\ d\theta$	Fully correct simplified integral with or without the 2π	A1

Question Number	Scheme	Notes	Marks
3(a)	$y = \text{arsech}\left(\frac{x}{2}\right) \Rightarrow \text{sech}\, y = \frac{x}{2}$ $\Rightarrow \frac{dx}{dy} = -2\,\text{sech}\, y \tanh y$	Takes "sech" of both sides and differentiates to obtain $\frac{dx}{dy} = k\,\text{sech}\, y \tanh y$ or equivalent.	M1
	$\Rightarrow \frac{dx}{dy} = -2\left(\frac{x}{2}\right)\sqrt{1-\left(\frac{x}{2}\right)^2}$ M1: Replaces $\text{sech}\, y$ with $\frac{x}{2}$ and $\tanh y$ with $\sqrt{1-\left(\frac{x}{2}\right)^2}$ A1: Correct equation involving $\frac{dx}{dy}$ or $\frac{dy}{dx}$ in any form in terms of x only.		M1A1
	$\Rightarrow \frac{dy}{dx} = \frac{-2}{x\sqrt{4-x^2}}$	Correct derivative in the required form or correct values for p and q.	A1
			(4)
(a) Way 2	$y = \text{arsech}\left(\frac{x}{2}\right) \Rightarrow \text{sech}\, y = \frac{x}{2}$ $\Rightarrow \cosh y = \frac{2}{x} \Rightarrow \sinh y \frac{dy}{dx} = -\frac{2}{x^2}$	Takes "sech" of both sides, changes to "cosh" and differentiates to obtain $\sinh y \frac{dy}{dx} = \frac{k}{x^2}$ or equivalent.	M1
	$\Rightarrow \frac{dy}{dx} = -\frac{2}{x^2 \sinh y} = -\frac{2}{x^2\sqrt{\left(\frac{2}{x}\right)^2 - 1}}$ M1: Replaces $\sinh y$ with $\sqrt{\left(\frac{2}{x}\right)^2 - 1}$ A1: Correct equation involving $\frac{dx}{dy}$ or $\frac{dy}{dx}$ in any form in terms of x only.		M1A1
	$\Rightarrow \frac{dy}{dx} = \frac{-2}{x\sqrt{4-x^2}}$	Correct derivative in the required form or correct values for p and q.	A1
(a) Way 3	$y = \text{arsech}\left(\frac{x}{2}\right) \Rightarrow y = \text{arcosh}\left(\frac{2}{x}\right)$ Changes to "arcosh" correctly. **Score this as the second M mark on EPEN.**		M1
	$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\left(\frac{2}{x}\right)^2 - 1}} \times -\frac{2}{x^2}$ M1: Differentiates to the form $\frac{k}{x^2\sqrt{\left(\frac{2}{x}\right)^2 - 1}}$ oe A1: Correct equation involving $\frac{dx}{dy}$ or $\frac{dy}{dx}$ in any form in terms of x only. **Score this as the first M mark and first A mark on EPEN.**		M1A1
	$\Rightarrow \frac{dy}{dx} = \frac{-2}{x\sqrt{4-x^2}}$	Correct derivative in the required form or correct values for p and q.	A1

(a) Way 4	$y=\text{arsech}\left(\frac{x}{2}\right)\Rightarrow \text{sech}\,y=\frac{x}{2}\Rightarrow\left(\frac{x}{2}\right)^2=\text{sech}^2 y\Rightarrow \tanh y=\sqrt{1-\left(\frac{x}{2}\right)^2}$ $\Rightarrow \text{sech}^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=-x\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$ Differentiates to $\text{sech}^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=kx\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$ or equivalent		M1
	$\Rightarrow \text{sech}^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=-x\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}\Rightarrow\frac{x^2}{4}\frac{\mathrm{d}y}{\mathrm{d}x}=-x\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{4}{x}\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$ M1: Replaces $\text{sech}^2 y$ with $\left(\frac{2}{x}\right)^2$ A1: Correct equation involving $\frac{\mathrm{d}x}{\mathrm{d}y}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$ in any form in terms of x only.		M1A1
	$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2}{x\sqrt{4-x^2}}$	Correct derivative in the required form or correct values for p and q.	A1
(a) Way 5	$y=\text{arsech}\left(\frac{x}{2}\right)\Rightarrow \text{sech}\,y=\frac{x}{2}\Rightarrow y=\text{artanh}\left(\sqrt{1-\left(\frac{x}{2}\right)^2}\right)$ Changes to "artanh" correctly. **Score this as the second M mark on EPEN.**		M1
	$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{1}{2}\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}}{1-\left(1-\frac{x^2}{4}\right)}\times-\frac{x}{2}$ M1: Differentiates to the form $\frac{kx\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}}{1-\left(1-\frac{x^2}{4}\right)}$ oe A1: Correct equation involving $\frac{\mathrm{d}x}{\mathrm{d}y}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$ in any form in terms of x only. **Score this as the first M mark and first A mark on EPEN.**		M1A1
	$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2}{x\sqrt{4-x^2}}$	Correct derivative in the required form or correct values for p and q.	A1

There may be other methods used.
If you are in any doubt if the method deserves any marks use Review.

(b)	$f(x)=\tanh^{-1}(x)+\text{sech}^{-1}\left(\frac{x}{2}\right)\Rightarrow f'(x)=\frac{1}{1-x^2}-\frac{2}{x\sqrt{4-x^2}}$ Correct f ′(*x*) following through their (a) of the form $\frac{p}{x\sqrt{q-x^2}}$ Also allow with "made up" *p* and *q* or the letters *p* and *q*.		B1ft
	$\frac{1}{1-x^2}-\frac{2}{x\sqrt{4-x^2}}=0\Rightarrow 2(1-x^2)=x\sqrt{4-x^2}\Rightarrow 4(1-x^2)^2=x^2(4-x^2)$ Sets $\frac{dy}{dx}=0$ with their (a) of the form $\frac{p}{x\sqrt{q-x^2}}$ and squares both sides to reach a quartic equation		M1
	$5x^4-12x^2+4=0$	Correct quartic	A1
	$5x^4-12x^2+4=0\Rightarrow x^2=2,\ 0.4$ $\Rightarrow x=\ldots$	Solves their quartic equation to obtain a value for x^2 and proceeds to a value for *x*. Apply usual rules for solving and check if necessary. Allow complex roots.	M1
	$x=\sqrt{\frac{2}{5}}$	Correct exact answer (allow equivalents e.g. $\frac{\sqrt{10}}{5}$). If any extra answers given score A0 e.g. $x=\pm\sqrt{\frac{2}{5}}$	A1
			(5)
			Total 9

Special case:

It is possible for a correct solution in (b) following a sign error in (a) e.g.

$$\frac{dy}{dx}=\frac{2}{x\sqrt{4-x^2}}$$

$$f(x)=\tanh^{-1}(x)+\text{sech}^{-1}\left(\frac{x}{2}\right)\Rightarrow f'(x)=\frac{1}{1-x^2}+\frac{2}{x\sqrt{4-x^2}}$$

$$\frac{1}{1-x^2}+\frac{2}{x\sqrt{4-x^2}}=0\Rightarrow 2(1-x^2)=-x\sqrt{4-x^2}\Rightarrow 4(1-x^2)^2=x^2(4-x^2)\text{ etc.}$$

This is likely to score M1M1A0A0 in (a) but allow full recovery in (b) if it leads to the correct answer.

Question Number	Scheme	Notes	Marks
4(a)	$\lambda=3\Rightarrow\lvert\mathbf{M}-3\mathbf{I}\rvert=\begin{vmatrix}3&k&2\\k&2&0\\2&0&4\end{vmatrix}=0\Rightarrow3(8)-k(4k)+2(-4)=0$ or e.g. $\lvert\mathbf{M}-\lambda\mathbf{I}\rvert=\begin{vmatrix}6-\lambda&k&2\\k&5-\lambda&0\\2&0&7-\lambda\end{vmatrix}=0$ $\Rightarrow(6-\lambda)(5-\lambda)(7-\lambda)-k\left(k(7-\lambda)\right)+2\left(0-2(5-\lambda)\right)=0\Rightarrow24-k(4k)-8=0$ Correct interpretation of 3 being an eigenvalue leading to the formation of **a quadratic equation in *k* only**. If the method for forming the determinant is not clear then look for at least 2 correct "components". NB rule of Sarrus gives $24-8-4k^2=0$		M1
	$\Rightarrow4k^2=16\Rightarrow k=\ldots$	Solves quadratic. **Depends on the first M.**	dM1
	$k=\pm2$	Correct values	A1
			(3)
(a) Way 2	$\begin{pmatrix}6&k&2\\k&5&0\\2&0&7\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3\begin{pmatrix}x\\y\\z\end{pmatrix}\Rightarrow\begin{matrix}6x+ky+2z=3x\\kx+5y=3y\\2x+7z=3z\end{matrix}$ $z=-\frac{1}{2}x,\ y=-\frac{1}{2}kx\Rightarrow6x-\frac{k^2x}{2}-x=3x\Rightarrow\frac{k^2}{2}=2$ Eliminates *z* and *y* and reaches a quadratic equation in *k* only		M1
	$\frac{k^2}{2}=2\Rightarrow k=\ldots$	Solves quadratic. **Depends on the first M.**	dM1
	$k=\pm2$	Correct values	A1
(b)	$k=-2\Rightarrow\lvert\mathbf{M}-\lambda\mathbf{I}\rvert=\begin{vmatrix}6-\lambda&-2&2\\-2&5-\lambda&0\\2&0&7-\lambda\end{vmatrix}$ $\Rightarrow(6-\lambda)(7-\lambda)(5-\lambda)+2(2\lambda-14)+2(2\lambda-10)=0$ Applies a value of *k* from (a) and a recognisable attempt at the characteristic equation (the "= 0" is not needed here). If the method is not clear then look for at least 2 correct "components".		M1
	$\Rightarrow\lambda^3-18\lambda^2+99\lambda-162=0\Rightarrow\lambda=\ldots$	Solves cubic. May use λ = 3 as a factor or calculator to solve. **Depends on the first mark.** Allow complex roots.	dM1
	$\lambda=6,\ 9\ (,3)$	Correct values. Allow to come from $k=2$	A1
			(3)

(c)	$\begin{pmatrix}6 & -2 & 2\\-2 & 5 & 0\\2 & 0 & 7\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3\begin{pmatrix}x\\y\\z\end{pmatrix}\Rightarrow \begin{matrix}6x-2y+2z=3x\\-2x+5y=3y\\2x+7z=3z\end{matrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\ldots$ or $\begin{pmatrix}3 & -2 & 2\\-2 & 2 & 0\\2 & 0 & 4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow \begin{matrix}6x-2y+2z=0\\-2x+5y=0\\2x+7z=0\end{matrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\ldots$ Correct strategy for finding the eigenvector using a value of *k* from (a) Note that the cross product of any 2 rows or columns of **M** – 3**I** gives an eigenvector		M1
	$p\begin{pmatrix}2\\2\\-1\end{pmatrix}$	Any correct eigenvector	A1
	$\frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}$	Any correct normalised eigenvector	A1
			(3)
			Total 9

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td>5(i)</td><td>$x^2-3x+5=\left(x-\frac{3}{2}\right)^2+\frac{11}{4}$</td><td>Correct completion of the square</td><td>B1</td></tr>
<tr><td></td><td colspan="2">$$\int\frac{1}{\sqrt{x^2-3x+5}}\,\mathrm{d}x=\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2+\frac{11}{4}}}\,\mathrm{d}x=\sinh^{-1}\frac{2x-3}{\sqrt{11}}(+c)$$
M1: Use of $\sinh^{-1}$
A1: Fully correct expression (condone omission of $+c$)
Allow equivalent correct expressions e.g. $\sinh^{-1}\frac{x-\frac{3}{2}}{\sqrt{\frac{11}{4}}}(+c)$, $\sinh^{-1}\frac{x-\frac{3}{2}}{\frac{\sqrt{11}}{2}}(+c)$
Allow equivalents for $\sinh^{-1}$ e.g. arsinh, arcsinh but not arsin or arcsin</td><td>M1A1</td></tr>
<tr><td></td><td colspan="2">You may see logarithmic forms for the answer:
e.g. $\ln\left(\frac{2x-3}{\sqrt{11}}+\sqrt{\left(\frac{2x-3}{\sqrt{11}}\right)^2+1}\right)$, $\ln\left(x-\frac{3}{2}+\sqrt{\left(x-\frac{3}{2}\right)^2+\frac{11}{4}}\right)$
but apply isw once a correct answer is seen.</td><td></td></tr>
<tr><td></td><td></td><td></td><td>(3)</td></tr>
<tr><td>(ii)</td><td>$63+4x-4x^2=-4\left(x^2-x-\frac{63}{4}\right)$
$=-4\left(\left(x-\frac{1}{2}\right)^2-\frac{64}{4}\right)$</td><td>Obtains $-4\left(\left(x-\frac{1}{2}\right)^2\pm\ldots\right)$ or
$-4\left(x-\frac{1}{2}\right)^2\pm\ldots$ or $\ldots-(2x-1)^2$</td><td>M1</td></tr>
<tr><td></td><td>$-4\left(\left(x-\frac{1}{2}\right)^2-16\right)$ or $64-4\left(x-\frac{1}{2}\right)^2$
or
$64-(2x-1)^2$</td><td>Correct completion of the square</td><td>A1</td></tr>
<tr><td></td><td colspan="2">$$\int\frac{1}{\sqrt{63+4x-4x^2}}\,\mathrm{d}x=\frac{1}{2}\sin^{-1}\left(\frac{2x-1}{8}\right)(+c)$$
M1: Use of $\sin^{-1}$
A1: Fully correct expression (condone omission of $+c$)
Allow equivalent correct expressions e.g. $\frac{1}{2}\sin^{-1}\frac{x-\frac{1}{2}}{4}(+c)$, $-\frac{1}{2}\sin^{-1}\frac{\frac{1}{2}-x}{4}(+c)$
Allow equivalents for $\sin^{-1}$ e.g. arsin, arcsin but not arsinh or arcsinh</td><td>M1A1</td></tr>
<tr><td></td><td></td><td></td><td>(4)</td></tr>
<tr><td></td><td colspan="2">In (ii) there are no marks for using $\int\frac{1}{\sqrt{63+4x-4x^2}}\,\mathrm{d}x=-\int\frac{1}{\sqrt{4x^2-63-4x}}\,\mathrm{d}x$
But if completion of square attempted first allow M1A1 e.g. for
$\int\frac{1}{\sqrt{63+4x-4x^2}}\,\mathrm{d}x=\int\frac{1}{\sqrt{64-(2x-1)^2}}\,\mathrm{d}x$ but then M0 for $=\int\frac{-1}{\sqrt{(2x-1)^2-64}}\,\mathrm{d}x$</td><td></td></tr>
<tr><td></td><td colspan="2"></td><td>Total 7</td></tr>
</table>

Question Number	Scheme	Notes	Marks
6(a)	$\int e^x \sin^n x\,dx = e^x \sin^n x - n\int e^x \sin^{n-1} x \cos x\,dx$ Applies integration by parts to obtain $\pm e^x \sin^n x \pm \alpha \int e^x \sin^{n-1} x \cos x\,dx$		M1
	$= e^x \sin^n x - n\left\{e^x \sin^{n-1} x \cos x - \int e^x\left((n-1)\sin^{n-2} x \cos^2 x - \sin^n x\right)dx\right\}$ M1: Applies integration by parts to $\pm\alpha\int e^x \sin^{n-1} x \cos x\,dx$ to obtain $\pm e^x \sin^{n-1} x \cos x \pm \int e^x\left(\alpha \sin^{n-2} x \cos^2 x - \beta \sin^n x\right)dx$ Or equivalent e.g. $\pm e^x \sin^{n-1} x \cos x \pm \int e^x\left(\alpha \sin^{n-2} x - \beta \sin^n x\right)dx$ (if Pythagoras applied first) A1: Fully correct expression for I_n from parts applied twice.		**d**M1A1
	$= e^x \sin^n x - n\left\{e^x \sin^{n-1} x \cos x - \int e^x\left((n-1)\sin^{n-2} x\left(1-\sin^2 x\right) - \sin^n x\right)dx\right\}$ Applies $\cos^2 x = 1 - \sin^2 x$		**d**M1
	$= e^x \sin^n x - n\left\{e^x \sin^{n-1} x \cos x - \int e^x\left((n-1)\sin^{n-2} x - (n-1)\sin^n x - \sin^n x\right)dx\right\}$ $= e^x \sin^n x - n\left\{e^x \sin^{n-1} x \cos x - \int e^x\left((n-1)\sin^{n-2} x - n\sin^n x\right)dx\right\}$ $= e^x \sin^n x - ne^x \sin^{n-1} x \cos x + n(n-1)I_{n-2} - n^2 I_n \Rightarrow I_n = \ldots$ Completes by introducing I_{n-2} and I_n and makes I_n the subject		**d**M1
	$I_n = \frac{e^x \sin^{n-1} x}{n^2+1}(\sin x - n\cos x) + \frac{n(n-1)}{n^2+1}I_{n-2}$ * Fully correct proof with no errors but allow e.g. the occasional missing "dx" but any clear errors must be recovered before final answer e.g. missing brackets.		A1*
			(6)

(b)	$$I_4=\frac{e^x\sin^3 x}{17}(\sin x-4\cos x)+\frac{12}{17}I_2$$ or $$I_2=\frac{e^x\sin x}{5}(\sin x-2\cos x)+\frac{2}{5}I_0$$ Applies the reduction formula once	M1
	$$=\frac{e^x\sin^3 x}{17}(\sin x-4\cos x)+\frac{12}{17}\left(\frac{e^x\sin x}{5}(\sin x-2\cos x)+\frac{2}{5}I_0\right)$$ $$=\frac{e^x\sin^3 x}{17}(\sin x-4\cos x)+\frac{12e^x\sin x}{85}(\sin x-2\cos x)+\frac{24}{85}e^x$$ Applies the reduction formula again and uses $I_0=\int e^x\,dx=e^x$ to obtain an expression in terms of x	M1
	$$\int_0^{\frac{\pi}{2}}e^x\sin^4 x\,dx=\left[\frac{e^x\sin^3 x}{17}(\sin x-4\cos x)+\frac{12e^x\sin x}{85}(\sin x-2\cos x)+\frac{24}{85}e^x\right]_0^{\frac{\pi}{2}}$$ $$=\frac{e^{\frac{\pi}{2}}}{17}+\frac{12e^{\frac{\pi}{2}}}{85}+\frac{24e^{\frac{\pi}{2}}}{85}-\frac{24}{85}$$ Uses the limits 0 and $\frac{\pi}{2}$ and subtracts. **Depends on both previous marks.**	**d**M1
	$$=\frac{41e^{\frac{\pi}{2}}}{85}-\frac{24}{85}$$ Correct expression or correct values e.g. $A=\ldots$, $B=\ldots$	A1
		(4)
	Note that the limits may be applied as they go e.g.: M1: $I_4=\frac{e^{\frac{\pi}{2}}}{17}(1-0)+\frac{12}{17}I_2$ $I_2=\frac{e^{\frac{\pi}{2}}}{5}(1-0)+\frac{2}{5}I_0$ $I_0=e^{\frac{\pi}{2}}-1$ M1M1: $I_4=\frac{e^{\frac{\pi}{2}}}{17}+\frac{12}{17}\left(\frac{e^{\frac{\pi}{2}}}{5}+\frac{2}{5}\left(e^{\frac{\pi}{2}}-1\right)\right)$ A1: $=\frac{41e^{\frac{\pi}{2}}}{85}-\frac{24}{85}$	
		Total 10

Question Number	Scheme	Notes	Marks
7(a)	$\frac{x-3}{4}=\frac{y-5}{-2}=\frac{z-4}{7}\Rightarrow \mathbf{r}=\begin{pmatrix}3\\5\\4\end{pmatrix}\pm\lambda\begin{pmatrix}4\\-2\\7\end{pmatrix}$	Converts to parametric form. "**r** =" is not required	M1
	$2x+4y-z=1$ $\Rightarrow 2(3+4\lambda)+4(5-2\lambda)-4-7\lambda=1$ $\Rightarrow \lambda=...(3)\Rightarrow P$ is ...	Correct strategy for finding P. Condone the use of $2x+4y-z=0$ for the plane equation.	M1
	$(15, -1, 25)$	Correct coordinates. Condone if given as a vector.	A1
			(3)
(a) Way 2	$\frac{x-3}{4}=\frac{y-5}{-2}\Rightarrow x=13-2y$	Uses the Cartesian equation to find x in terms of y	M1
	$2x+4y-z=1\Rightarrow 26-4y+4y-z=1$ $\Rightarrow z=..., x=..., y=...$	Correct strategy for finding P. Condone the use of $2x+4y-z=0$ for the plane equation.	M1
	$(15, -1, 25)$	Correct coordinates. Condone if given as a vector.	A1
(b)	$\begin{pmatrix}4\\-2\\7\end{pmatrix}\bullet\begin{pmatrix}2\\4\\-1\end{pmatrix}=8-8-7=-7$	Applies the scalar product between the direction of l_1 and the normal to the plane	M1
	Examples: $\phi=\cos^{-1}\frac{\pm 7}{\sqrt{69}\sqrt{21}}=...\quad \phi=\sin^{-1}\frac{\pm 7}{\sqrt{69}\sqrt{21}}=...$ Attempts to find a relevant angle in degrees or radians. **Depends on the first method mark.**		**d**M1
	$\theta=10.6°$	Allow awrt 10.6 but do **not** isw and mark the final answer. For reference $\theta=10.5965654°$	A1
			(3)
(b) Way 2	$\begin{pmatrix}4\\-2\\7\end{pmatrix}\times\begin{pmatrix}2\\4\\-1\end{pmatrix}=\begin{pmatrix}26\\-18\\-20\end{pmatrix}$	Attempts vector product of normal to Π and direction of l_1	M1
	$\sqrt{26^2+18^2+20^2}=\sqrt{21}\sqrt{69}\sin\alpha$ $\sin\alpha=\frac{10\sqrt{46}}{69}\Rightarrow\alpha=...$	Attempts to find a relevant angle. **Depends on the first method mark.**	**d**M1
	$\theta=10.6°$	Allow awrt 10.6 but do **not** isw and mark the final answer. For reference $\theta=10.5965654°$	A1

(c)	$\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 4 & -1\\ 4 & -2 & 7\end{vmatrix}=\begin{pmatrix}26\\-18\\-20\end{pmatrix}$	Attempts vector product of normal to Π and direction of l_1. If no method is seen expect at least 2 correct components.	M1
	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 13 & -9 & -10\\ 2 & 4 & -1\end{vmatrix}=\begin{pmatrix}49\\-7\\70\end{pmatrix}$	Attempts vector product of "**a**" with normal to Π to find direction of l_2	M1
		Correct direction for l_2	A1
	$\mathbf{r}=\begin{pmatrix}15\\-1\\25\end{pmatrix}+\mu\begin{pmatrix}7\\-1\\10\end{pmatrix}$	Depends on both previous M marks Attempts vector equation using their direction vector and their P	ddM1
		Correct equation or any equivalent correct vector equation	A1
			(5)
(c) Way 2	$\lambda=1\Rightarrow(7,\ 3,\ 11)$ lies on l_1 $\mathbf{r}=\begin{pmatrix}7\\3\\11\end{pmatrix}+t\begin{pmatrix}2\\4\\-1\end{pmatrix}$ $\Rightarrow 2(7+2t)+4(3+4t)-11+t=1$ $t=-\frac{2}{3}\Rightarrow\left(\frac{17}{3},\ \frac{1}{3},\ \frac{35}{3}\right)$ is on l_2	Complete method to find a point on l_2	M1
	Direction of l_2 is $\begin{pmatrix}15\\-1\\25\end{pmatrix}-\frac{1}{3}\begin{pmatrix}17\\1\\35\end{pmatrix}=\frac{1}{3}\begin{pmatrix}28\\-4\\40\end{pmatrix}$	Uses their point and their P to find direction of l_2	M1
		Correct direction for l_2	A1
	$\mathbf{r}=\begin{pmatrix}15\\-1\\25\end{pmatrix}+\mu\begin{pmatrix}7\\-1\\10\end{pmatrix}$	Attempts vector equation using their direction vector and their point on l_2	ddM1
		Correct equation or any equivalent correct vector equation. Must have $\mathbf{r}=$ and not e.g. $l_2=\ldots$	A1
(c) Way 3	Normal to plane from l_1 $\mathbf{r}=\begin{pmatrix}3\\5\\4\end{pmatrix}+t\begin{pmatrix}2\\4\\-1\end{pmatrix}$ $\Rightarrow 2(3+2t)+4(5+4t)-(4-t)=1$ $t=-1\Rightarrow(1,\ 1,\ 5)$ is on l_2	Complete method to find a point on l_2	M1
	Direction of l_2 is $\begin{pmatrix}15\\-1\\25\end{pmatrix}-\begin{pmatrix}1\\1\\5\end{pmatrix}=\begin{pmatrix}14\\-2\\20\end{pmatrix}$	Uses their point and their P to find direction of l_2	M1
		Correct direction for l_2	A1
	$\mathbf{r}=\begin{pmatrix}1\\1\\5\end{pmatrix}+\mu\begin{pmatrix}7\\-1\\10\end{pmatrix}$	Attempts vector equation using their direction vector and their point on l_2	ddM1
		Correct equation or any equivalent correct vector equation. Must have $\mathbf{r}=$ and not e.g. $l_2=\ldots$	A1
			Total 11

Question Number	Scheme	Notes	Marks
8(a)	$b^2 = a^2\left(1-e^2\right) \Rightarrow 4 = 9\left(1-e^2\right) \Rightarrow e = ...$ or e.g. $e = \sqrt{1-\frac{b^2}{a^2}} \Rightarrow e = ...$	Uses a correct formula with a and b correctly placed to find a value for e	M1
	$e = \frac{\sqrt{5}}{3}$	Correct value (or equivalent) $e = \pm\frac{\sqrt{5}}{3}$ scores A0	A1
			(2)
(b)(i)	$(\pm ae,\ 0) = \left(\pm\sqrt{5},\ 0\right)$ or $\left(\pm 3\frac{\sqrt{5}}{3},\ 0\right)$ Correct foci. Must be coordinates but allow unsimplified and isw if necessary. Follow through their e so allow for ($\pm$ 3$\times$their e, 0)		B1ft
(ii)	$x = \pm\frac{a}{e} = \pm\frac{9}{\sqrt{5}}$ or $x = \pm\frac{3}{\frac{\sqrt{5}}{3}}$ Correct directrices. Must be equations but allow unsimplified and isw if necessary. Follow through their e so allow for $x = \pm 3$/their e		B1ft
			(2)
	Special case: Use of a^2 for a and b^2 for b **consistently** scores M0A0 in (a) and B1ft B1ft in (b) This gives $e = \frac{\sqrt{65}}{9}$, $\left(\pm\sqrt{65},\ 0\right)$, $x = \pm\frac{81}{\sqrt{65}}$		
(c)	$\frac{dx}{d\theta} = -3\sin\theta,\ \frac{dy}{d\theta} = 2\cos\theta$ or $\frac{2x}{9} + \frac{2y}{4}\frac{dy}{dx} = 0$ or $y = \left(4-\frac{4x^2}{9}\right)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = -\frac{4x}{9}\left(4-\frac{4x^2}{9}\right)^{-\frac{1}{2}}$ $\Rightarrow \frac{dy}{dx} = ...\left(= \frac{2\cos\theta}{-3\sin\theta}\right)$	Correct strategy for the gradient of l in terms of θ. Allow $\frac{dy}{dx} = \frac{2\cos\theta}{-3\sin\theta}$ to be stated.	M1
	$y - 2\sin\theta = \frac{2\cos\theta}{-3\sin\theta}\left(x - 3\cos\theta\right)$	Correct straight line method (any complete method). Finding the equation of the normal is M0.	M1
	$-3y\sin\theta + 6\sin^2\theta = 2x\cos\theta - 6\cos^2\theta$ $2x\cos\theta + 3y\sin\theta = 6$*	Cso with at least one intermediate line of working	A1*
			(3)

(d)	$l_2: \quad y = \dfrac{3\sin\theta}{2\cos\theta}x$	Correct equation for l_2	B1
	$2x\cos\theta + 3y\sin\theta = 6,\ y = \dfrac{3\sin\theta}{2\cos\theta}x$ $\Rightarrow x = \ldots,\ y = \ldots$	Complete method for Q	M1
	$Q: \left(\dfrac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}, \dfrac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}\right)$ Correct coordinates. Allow as $x = \ldots,\ y = \ldots$ and allow equivalent correct expressions as long as they are single fractions e.g. $x = \dfrac{12\cos\theta}{4 + 5\sin^2\theta}\quad y = \dfrac{18\sin\theta}{4 + 5\sin^2\theta},\quad x = \dfrac{12\cos\theta}{9 - 5\cos^2\theta}\quad y = \dfrac{18\sin\theta}{9 - 5\cos^2\theta}$		A1
			(3)

(e)	At Q, $\dfrac{y}{x}=\dfrac{3}{2}\tan\theta$	Uses their coordinates of Q to attempt an equation connecting x, y and θ or states or uses the equation found in (d)	M1
	$x=\dfrac{12\cos\theta}{4\cos^2\theta+9\sin^2\theta}=\dfrac{12\sec\theta}{4+9\tan^2\theta}\Rightarrow x^2=\dfrac{144\sec^2\theta}{\left(4+9\tan^2\theta\right)^2}=\dfrac{144\left(1+\dfrac{4y^2}{9x^2}\right)}{\left(4+9\times\dfrac{4y^2}{9x^2}\right)^2}$ or $y=\dfrac{18\sin\theta}{4\cos^2\theta+9\sin^2\theta}=\dfrac{12\sec\theta\tan\theta}{4+9\tan^2\theta}$ $\Rightarrow y^2=\dfrac{324\sec^2\theta\tan^2\theta}{\left(4+9\tan^2\theta\right)^2}=\dfrac{324\left(1+\dfrac{4y^2}{9x^2}\right)\dfrac{4y^2}{9x^2}}{\left(4+9\times\dfrac{4y^2}{9x^2}\right)^2}$ Eliminates θ **Depends on the first mark.**		**dM1**
	$\Rightarrow x^2=\dfrac{x^2\left(9x^2+4y^2\right)}{\left(x^2+y^2\right)^2}\Rightarrow\left(x^2+y^2\right)^2=9x^2+4y^2$ or $\Rightarrow 9\times16x^2y^2\left(1+\dfrac{y^2}{x^2}\right)^2=4\times18^2\left(1+\dfrac{4y^2}{9x^2}\right)\Rightarrow\left(x^2+y^2\right)^2=9x^2+4y^2$ Correct equation or correct values for α and β.		A1
			(3)
(e) Way 2	$x=\dfrac{12\cos\theta}{4+5\sin^2\theta}\quad y=\dfrac{18\sin\theta}{4+5\sin^2\theta}\Rightarrow\left(x^2+y^2\right)^2=\left(\dfrac{144\cos^2\theta+324\sin^2\theta}{\left(4+5\sin^2\theta\right)^2}\right)^2$ Uses their Q to obtain an expression for $\left(x^2+y^2\right)^2$ in terms of θ		M1
	$\left(\dfrac{144\cos^2\theta+324\sin^2\theta}{\left(4+5\sin^2\theta\right)^2}\right)^2=\left(\dfrac{144+180\sin^2\theta}{\left(4+5\sin^2\theta\right)^2}\right)^2=\left(\dfrac{36\left(4+5\sin^2\theta\right)}{\left(4+5\sin^2\theta\right)^2}\right)^2=\dfrac{1296}{\left(4+5\sin^2\theta\right)^2}$ $\dfrac{1296}{\left(4+5\sin^2\theta\right)^2}=\alpha x^2+\beta y^2=\alpha\dfrac{144\cos^2\theta}{\left(4+5\sin^2\theta\right)^2}+\beta\dfrac{324\sin^2\theta}{\left(4+5\sin^2\theta\right)^2}\Rightarrow\alpha=...,\beta=...$ Substitutes into the given answer and solves for α and β **Depends on the first mark.**		**dM1**
	$\left(x^2+y^2\right)^2=9x^2+4y^2$	Correct expression or correct values for α and β.	A1
			Total 13

Question Number	Scheme	Notes	Marks
1(a)	$\left(\cosh A\cosh B+\sinh A\sinh B=\right)\left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right)+\left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$ $=\frac{e^{A+B}+e^{A-B}+e^{B-A}+e^{-A-B}+e^{A+B}-e^{A-B}-e^{B-A}+e^{-A-B}}{4}$ Expresses the lhs in terms of exponentials correctly, combines terms and combines fractions with common denominator (Brackets not needed due to fraction lines)		M1
	$=\frac{2e^{A+B}+2e^{-(A+B)}}{4}=\frac{e^{A+B}+e^{-(A+B)}}{2}=\cosh(A+B)$* Fully correct proof with no errors		A1*
			(2)
(b)	$\cosh(x+\ln 2)=\cosh x\cosh(\ln 2)+\sinh x\sinh(\ln 2)$ $=\left(\frac{2+\frac{1}{2}}{2}\right)\cosh x+\left(\frac{2-\frac{1}{2}}{2}\right)\sinh x$ Applies the result from part (a) and evaluates both cosh(ln2) and sinh(ln2) **Use of (a) must be seen**		M1
	$\frac{5}{4}\cosh x+\frac{3}{4}\sinh x=5\sinh x$ $\Rightarrow\frac{5}{4}\cosh x=\frac{17}{4}\sinh x$	Collects terms and reaches $a\cosh x=b\sinh x$ oe Depends on the first M mark	dM1
	$5\cosh x=17\sinh x$ oe	Correct equation	A1
	$x=\frac{1}{2}\ln\left(\frac{1+\frac{5}{17}}{1-\frac{5}{17}}\right)$ Or $\frac{e^{2x}-1}{e^{2x}+1}=\frac{5}{17}\Rightarrow x=\ldots$	Moves to $\tanh x$ and uses the correct logarithmic form for artanh$x$ or reverts to exponential forms and solves for x Depends on both M marks	ddM1
	$x=\frac{1}{2}\ln\left(\frac{11}{6}\right)$	Cao $\left(\text{Accept integer multiples of }\frac{11}{6}\right)$	A1
			(5)
			Total 7

Way 2			
(b)	$\cosh(x+\ln 2)=\cosh x\cosh(\ln 2)+\sinh x\sinh(\ln 2)$ $=\left(\frac{2+\frac{1}{2}}{2}\right)\cosh x+\left(\frac{2-\frac{1}{2}}{2}\right)\sinh x$ Applies the result from part (a) and evaluates both cosh(ln2) and sinh(ln2) **Use of (a) must be seen**		M1
	$\Rightarrow 5\cosh x=17\sinh x$ dM1: Collects terms and reaches an equation of form $A\cosh x=B\sinh x$ A1: Correct equation		dM1A1
	$5\left(\frac{e^x+e^{-x}}{2}\right)=17\left(\frac{e^x-e^{-x}}{2}\right)$		
	$12e^x=22e^{-x}\Rightarrow e^{2x}=\frac{22}{6}\Rightarrow x=...$	Changes to exponentials (correct forms) And solves for x	ddM1
	$x=\frac{1}{2}\ln\left(\frac{11}{6}\right)$	Cao $\left(\text{Accept integer multiples of } \frac{11}{6}\right)$	A1
Way 3			
	$\cosh(x+\ln 2)=\cosh x\cosh(\ln 2)+\sinh x\sinh(\ln 2)$ $\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^{\ln 2}+e^{-\ln 2}}{2}\right)+\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^{\ln 2}-e^{-\ln 2}}{2}\right)=5\left(\frac{e^x-e^{-x}}{2}\right)$ Applies the result from part (a) and uses the exponential forms of the hyperbolic functions. **Use of (a) must be seen**		M1
	eg $5e^x+5e^{-x}=17e^x-17e^{-x}$ oe	Evaluates $e^{\ln 2}$ and $e^{-\ln 2}$ and starts to collect terms	dM1
	$12e^{2x}=22\Rightarrow e^{2x}=\frac{11}{6}$	Correct value for e^{2x}	A1
	$x=...$	Solves for x	ddM1
	$x=\frac{1}{2}\ln\left(\frac{11}{6}\right)$	Cao $\left(\text{Accept integer multiples of } \frac{11}{6}\right)$	A1

NB: Squaring and obtaining a value for sinh*x* or cosh*x* introduces extra answers. If these extra answers are then eliminated M1A1 is available but if no attempt at elimination is made award M0A0

Question Number	Scheme	Notes	Marks
2(i)	Throughout both parts of this question do not penalise omission of $\mathrm{d}x$ or $\mathrm{d}\theta$		
	$5+4x-x^2=9-(x-2)^2$ oe	Correct completion of the square Any correct result	B1
	$\int \frac{1}{\sqrt{5+4x-x^2}}\,\mathrm{d}x=\int \frac{1}{\sqrt{9-(x-2)^2}}\,\mathrm{d}x=\sin^{-1}\left(\frac{x-2}{3}\right)(+c)$ M1: Obtains $k\sin^{-1}\mathrm{f}(x)$ A1: Correct integration (+ c not needed)		M1A1
			(3)
(ii)	$x=6\Rightarrow\theta=\frac{\pi}{3}$ $x=2\sqrt{3}\Rightarrow\theta=\frac{\pi}{6}$	Correct θ limits in radians	B1
	$\int \frac{18}{(x^2-9)^{\frac{3}{2}}}\,\mathrm{d}x=\int \frac{18\times 3\sec\theta\tan\theta}{(9\sec^2\theta-9)^{\frac{3}{2}}}\,\mathrm{d}\theta$ M1: For $\int \frac{18}{\left((3\sec\theta)^2-9\right)^{\frac{3}{2}}}\times\left(\text{their }\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)\mathrm{d}\theta$		M1
	$\int \frac{54\sec\theta\tan\theta}{(9\sec^2\theta-9)^{\frac{3}{2}}}\,\mathrm{d}\theta=54\int \frac{\sec\theta\tan\theta}{27\tan^3\theta}\,\mathrm{d}\theta=2\int \frac{\sin\theta\cos^3\theta}{\cos^2\theta\sin^3\theta}\,\mathrm{d}\theta$ $2\int \frac{\cos\theta}{\sin^2\theta}\,\mathrm{d}\theta$ oe eg $2\int \frac{\sec\theta}{\tan^2\theta}\,\mathrm{d}\theta$ Correct simplified integral		A1
	$2\int \frac{\cos\theta}{\sin^2\theta}\,\mathrm{d}\theta=2\int \operatorname{cosec}\theta\cot\theta\,\mathrm{d}\theta=-2\operatorname{cosec}\theta(+c)$ Obtains $k\operatorname{cosec}\theta(+c)$		M1
	$[-2\operatorname{cosec}\theta]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=-2\operatorname{cosec}\frac{\pi}{3}+2\operatorname{cosec}\frac{\pi}{6}$	Uses changed limits correctly. **Depends on all previous method marks.**	dM1
	$=4-\frac{4}{3}\sqrt{3}$	Cao Allow these 2 marks if limits have been given in degrees	A1
			(6)
			Total 9

ALT	For B1 and final dM1A1 of (ii)	
	dM1: Reverse the substitution A1: Correct reversed result A1: **enter as B1 on e-PEN** Correct final answer	

Question Number	Scheme	Notes	Marks
3(a)	3	Correct value seen in (a)	B1
			(1)
(b)	$\begin{pmatrix}-2 & 5 & 0\\ 5 & 1 & -3\\ 0 & -3 & 6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}8x\\8y\\8z\end{pmatrix}\Rightarrow\begin{matrix}-2x+5y=8x\\5x+y-3z=8y\\-3y+6z=8z\end{matrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=...$ Correct method for the eigenvector (making a variable equal to 0 is not a correct method)		M1
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}$	Any correct eigenvector	A1
			(2)
(c)	$\lvert\mathbf{M}-\lambda\mathbf{I}\rvert=\begin{vmatrix}-2-\lambda & 5 & 0\\ 5 & 1-\lambda & -3\\ 0 & -3 & 6-\lambda\end{vmatrix}=0$ $\Rightarrow(-2-\lambda)\left[(1-\lambda)(6-\lambda)-9\right]-5\left[5(6-\lambda)\right]=0\Rightarrow\lambda=...$ NB CE is $\lambda^3-5\lambda^2-42\lambda+144=0$ but may only find the constant term		M1
	$\lambda=-6$	Correct third eigenvalue The work for these 2 marks may be seen in (a) – award them Correct third eigenvalue by a different method – send to review	A1
	$\mathbf{D}=\begin{pmatrix}3 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & -6\end{pmatrix}$	Correct **D** following through their third eigenvalue	A1ft
	$\begin{pmatrix}-2 & 5 & 0\\ 5 & 1 & -3\\ 0 & -3 & 6\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-6x\\-6y\\-6z\end{pmatrix}\Rightarrow\begin{matrix}-2x+5y=-6x\\5x+y-3z=-6y\\-3y+6z=-6z\end{matrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\left(\begin{pmatrix}-5\\4\\1\end{pmatrix}\right)$ Correct strategy for 3rd eigenvector		M1
	$\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & -\frac{5}{\sqrt{42}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{4}{\sqrt{42}}\\ \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{42}}\end{pmatrix}$	Fully correct matrix consistent with their **D** May have $\frac{\sqrt{3}}{3}$ etc	A1
			(5)
			Total 8

Question Number	Scheme	Notes	Marks
4.	$y = \text{artanh}\left(\dfrac{\cos x + a}{\cos x - a}\right)$		
	$\dfrac{dy}{dx} = \dfrac{1}{1-\left(\dfrac{\cos x + a}{\cos x - a}\right)^2} \times \dfrac{(\cos x - a)\times -\sin x - (\cos x + a)\times -\sin x}{(\cos x - a)^2}$ or $\dfrac{dy}{dx} = \dfrac{1}{1-\left(\dfrac{\cos x + a}{\cos x - a}\right)^2} \times \left(-\sin x \times (\cos x - a)^{-1} + (\cos x + a)\times \sin x (\cos x - a)^{-2}\right)$ M1: Correct method for the derivative. This requires $\dfrac{1}{1-\left(\dfrac{\cos x + a}{\cos x - a}\right)^2} \times$ An attempt at the quotient (or product) rule. A1: Correct derivative in any form		M1A1
	$= \dfrac{(\cos x - a)^2}{(\cos x - a)^2 - (\cos x + a)^2} \times \dfrac{2a\sin x}{(\cos x - a)^2} = \dfrac{2a\sin x}{-4a\cos x} = \ldots$ Uses correct processing to reach $\lambda \dfrac{\sin x}{\cos x}$ or $\lambda \tan x$ **Depends on the first method mark.**		dM1
	$= -\dfrac{1}{2}\tan x$	cso	A1 **(4)**
Way 2	$y = \text{artanh}\left(\dfrac{\cos x + a}{\cos x - a}\right) \Rightarrow \tanh y = \dfrac{\cos x + a}{\cos x - a} \Rightarrow \text{sech}^2 y \dfrac{dy}{dx} = \dfrac{2a\sin x}{(\cos x - a)^2}$ Takes tanh of both sides, obtains $\text{sech}^2 y \dfrac{dy}{dx} =$ an attempt at the quotient or product rule		M1
	$\dfrac{dy}{dx} = \dfrac{1}{1-\left(\dfrac{\cos x + a}{\cos x - a}\right)^2} \times \dfrac{2a\sin x}{(\cos x - a)^2}$ Correct derivative in any form		A1
	$= \dfrac{(\cos x - a)^2}{(\cos x - a)^2 - (\cos x + a)^2} \times \dfrac{2a\sin x}{(\cos x - a)^2} = \dfrac{2a\sin x}{-4a\cos x} = \ldots$ Uses correct processing to reach $\lambda \dfrac{\sin x}{\cos x}$ or $\lambda \tan x$ **Depends on the first method mark.**		dM1
	$= -\dfrac{1}{2}\tan x$	cso	A1 **(4)**

Way 3	Uses substitution $u=\frac{\cos x+a}{\cos x-a}$, obtains $\frac{du}{dx}\left(=\frac{2a\sin x}{(\cos x-a)^2}\right)$ by quotient rule and $\frac{dy}{du}\left(=\frac{1}{1-u^2}\right)$ followed by chain rule to obtain $\frac{dy}{dx}=\frac{1}{1-\left(\frac{\cos x+a}{\cos x-a}\right)^2}\times\frac{2a\sin x}{(\cos x-a)^2}$		M1
	Correct derivative in any form		A1
	Uses correct processing to reach $\lambda\frac{\sin x}{\cos x}$ or $\lambda\tan x$ **Depends on the first method mark.**		dM1
	$=-\frac{1}{2}\tan x$	cso	A1 **(4)**
			Total 4
Way 4	$y=\frac{1}{2}\ln\left(\frac{1+\frac{\cos x+a}{\cos x-a}}{1-\frac{\cos x+a}{\cos x-a}}\right)=\frac{1}{2}\ln\left(-\frac{\cos x}{a}\right)$ $\frac{dy}{dx}=\frac{1}{2}\times\frac{1}{-\frac{\cos x}{a}}\times\left(\frac{\sin x}{a}\right)$	M1: Converts to correct ln form and uses chain rule to differentiate A1: Correct derivative in any form	M1A1
	Uses correct processing to reach $\lambda\frac{\sin x}{\cos x}$ or $\lambda\tan x$ **Depends on the first method mark.**		dM1
	$=-\frac{1}{2}\tan x$	cso	A1
			(4)

Question Number	Scheme	Notes	Marks
5	$x = 4e^{\frac{1}{2}t}, \quad y = e^t - t \qquad 0 \leqslant t \leqslant 4$		
	$\frac{dx}{dt} = 2e^{\frac{1}{2}t}, \quad \frac{dy}{dt} = e^t - 1$	Correct derivatives	B1
	NB: Allow missing dt in the following integration work		
	$S = (2\pi)\int y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}(dt) = (2\pi)\int (e^t - t)\sqrt{\left(4e^{\frac{1}{2}t}\right)^2 + \left(e^t - t\right)^2}\ (dt)$ $\left(= (2\pi)\int (e^t - t)\sqrt{4e^t + e^{2t} - 2e^t + 1}(dt)\right)$ Applies the surface area formula with or w/o the 2π		M1
	$= (2\pi)\int (e^t - t)(e^t + 1)(dt)$	Correct simplified integral Brackets must be present unless implied by subsequent work but award by implication if $(2\pi)\int (e^{2t} + e^t - te^t - t)(dt)$ is seen	A1
	$= (2\pi)\int (e^t - t)(e^t + 1)(dt) = (2\pi)\int (e^{2t} + e^t - te^t - t)(dt)$ $= (2\pi)\left[\frac{1}{2}e^{2t} + e^t - te^t + e^t - \frac{1}{2}t^2\right]$ B1: For $\int te^t\ dt = te^t - e^t\ (+c)$ A1: Fully correct integration (the integration may be shown as 2 separate parts and score B1A1 if both parts correct)		B1A1
	$= 2\pi\left[\frac{1}{2}e^{2t} + 2e^t - te^t - \frac{1}{2}t^2\right]_0^4 = 2\pi\left\{\left(\frac{1}{2}e^8 + 2e^4 - 4e^4 - 8\right) - \left(\frac{1}{2} + 2\right)\right\}$ Applies the limits 0 and 4 Must include 2π now. If 2 integrals have been used limits must be applied to both and the results added Depends on the first M mark (and some valid integration)		dM1
	$\pi\left(e^8 - 4e^4 - 21\right)$	Cao	A1
			(7)
			Total 7

Question Number	Scheme	Notes	Marks
6(a)	$\mathbf{A}=\begin{pmatrix} x & 1 & 3 \\ 2 & 4 & x \\ -4 & -2 & -1 \end{pmatrix}$		
	NB: Work for (a) can only be awarded in (a)		
	$\lvert\mathbf{A}\rvert = x(-4+2x)-(-2+4x)+3(-4+16)$	Correct determinant attempt (expand by any row or column) or use the Rule of Sarrus (send to review if unsure) Sign errors allowed **only within the brackets**	M1
	$=2x^2-8x+38$	Correct simplified determinant	A1
	$2x^2-8x+38=2(x-2)^2+30$ or $\frac{\mathrm{d}}{\mathrm{d}x}(2x^2-8x+38)=4x-8=0 \Rightarrow x=2$ $\Rightarrow 2x^2-8x+38=...$ or $b^2-4ac=64-4\times2\times38=...$	Starts the process of showing det $\mathbf{A}\neq 0$ E.g. Completes the square, finds the minimum point or finds discriminant May find discriminant of $x^2-4x+19=...$	M1
	$2x^2-8x+38 \geqslant 30$ or $b^2-4ac<0$ Therefore det $\mathbf{A}\neq 0$ which means $\mathbf{A}$ is non-singular	Appropriate reasoning for their chosen method and a conclusion stating that $\mathbf{A}$ is non-singular. **All 3 previous marks needed** (No need to evaluate a discriminant, so ISW slips in calculation provided $64-4\times2\times38=...$or $16-4\times19=...$ seen	A1cso
			(4)
(b)	$\begin{pmatrix} x & 1 & 3 \\ 2 & 4 & x \\ -4 & -2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -4+2x & -2+4x & -4+16 \\ -1+6 & -x+12 & -2x+4 \\ x-12 & x^2-6 & 4x-2 \end{pmatrix} \rightarrow \begin{pmatrix} -4+2x & 2-4x & 12 \\ -5 & -x+12 & 2x-4 \\ x-12 & -x^2+6 & 4x-2 \end{pmatrix}$ M1: Applies the correct method to reach at least a matrix of cofactors 2 correct rows or 2 correct columns needed A1: Correct cofactor matrix		M1A1
	$\begin{pmatrix} -4+2x & 2-4x & 12 \\ -5 & -x+12 & 2x-4 \\ x-12 & -x^2+6 & 4x-2 \end{pmatrix} \rightarrow \begin{pmatrix} -4+2x & -5 & x-12 \\ 2-4x & -x+12 & -x^2+6 \\ 12 & 2x-4 & 4x-2 \end{pmatrix}$ $\mathbf{A}^{-1}=\frac{1}{2x^2-8x+38}\begin{pmatrix} -4+2x & -5 & x-12 \\ 2-4x & -x+12 & -x^2+6 \\ 12 & 2x-4 & 4x-2 \end{pmatrix}$ dM1: Transposes and divides by their determinant.		dM1A1

	If their original determinant has been divided by 2 (acceptable for (a)) and then used here it is **not** their determinant and so scores dM0 2 correct rows or 2 correct columns needed from their previous matrix **Depends on previous method mark**. A1: Correct matrix		
			(4)
			Total 8

Question Number	Scheme	Notes	Marks
7.	$I_n = \int \frac{x^n}{\sqrt{10-x^2}}\,dx \qquad n \in \mathbb{N},\ \lvert x\rvert < \sqrt{10}$		
(a)	$I_n = \int \frac{x^n}{\sqrt{10-x^2}}\,dx = \int \frac{x^{n-1}\times x}{\sqrt{10-x^2}}\,dx$	Writes x^n as $x \times x^{n-1}$	M1
	$\int \frac{x^{n-1}\times x}{\sqrt{10-x^2}}\,dx = -x^{n-1}\left(10-x^2\right)^{\frac{1}{2}} + (n-1)\int x^{n-2}\left(10-x^2\right)^{\frac{1}{2}}\,dx$ **d**M1: Uses integration by parts to obtain $\int \frac{x^{n-1}\times x}{\sqrt{10-x^2}}\,dx = \alpha x^{n-1}\left(10-x^2\right)^{\frac{1}{2}} + \beta\int x^{n-2}\left(10-x^2\right)^{\frac{1}{2}}\,dx$ A1: Correct expression		**d**M1A1
	$= \ldots + (n-1)\int x^{n-2}\left(10-x^2\right)\left(10-x^2\right)^{-\frac{1}{2}}\,dx$ $= \ldots + 10(n-1)\int x^{n-2}\left(10-x^2\right)^{-\frac{1}{2}}\,dx - (n-1)\int x^{n}\left(10-x^2\right)^{-\frac{1}{2}}\,dx$ Applies $\left(10-x^2\right)^{\frac{1}{2}} = \left(10-x^2\right)\left(10-x^2\right)^{-\frac{1}{2}}$ and splits into 2 integrals		**d**M1
	$= \ldots + 10(n-1)I_{n-2} - (n-1)I_n \Rightarrow nI_n$	Introduces I_{n-2} and I_n and makes progress to the given result	**d**M1
	$nI_n = 10(n-1)I_{n-2} - x^{n-1}\left(10-x^2\right)^{\frac{1}{2}}$ * Fully correct proof with no errors (recovery of missing brackets counts as an error) as does missing dx		A1*
			(6)
(b)	$I_1 = \int_0^1 \frac{x}{\sqrt{10-x^2}}\,dx = \left[-\left(10-x^2\right)^{\frac{1}{2}}\right]_0^1 \left(= -3+\sqrt{10}\right)$ Correct method for I_1 Limits can be substituted later		M1
	$5I_5 = 10\times 4I_3 + \ldots$	Applies the reduction formula at least once Allow with 3 or $\left[-x^4\left(10-x^2\right)^{\frac{1}{2}}\right]_0^1$	M1
	$I_5 = 8I_3 - \frac{3}{5} = 8\left(\frac{20}{3}I_1 - 1\right) - \frac{3}{5} = \frac{160}{3}I_1 - \frac{43}{5}$ $I_5 = \frac{160}{3}\left(\sqrt{10}-3\right) - \frac{43}{5}$ Completes the process using their I_1 to obtain a numerical value for I_5 Limits must now be substituted		M1
	$= \frac{1}{15}\left(800\sqrt{10} - 2529\right)$	Cao	A1
			(4)
			Total 10

Question Number	Scheme	Notes	Marks
8(a)	$(\mathbf{r}=)\begin{pmatrix}-4\\-5\\3\end{pmatrix}+t\begin{pmatrix}3\\4\\-1\end{pmatrix}$	Forms the parametric form of the line	M1
	$3(3t-4)+4(4t-5)-(3-t)=17$ $\Rightarrow t=(2)$	Substitutes the parametric form for the line into the plane equation and solves for "t". **Depends on the first mark.**	dM1
	$\begin{pmatrix}-4\\-5\\3\end{pmatrix}+"2"\begin{pmatrix}3\\4\\-1\end{pmatrix}$	Uses their value of t correctly to find Q. **Depends on the previous mark.**	dM1
	$(2, 3, 1)$	Correct coordinates Accept if written as a column vector but not with **i, j, k**	A1 **(4)**
Way 2	$\frac{x+4}{3}=\frac{y+5}{4}=\frac{z-3}{-1}$ eg $x=\mathrm{f}(y)\ \ z=\mathrm{g}(y)$	Forms the Cartesian equation of the line, rearranges twice to get 2 of x, y, z as functions of the third	M1
		Substitutes these into the plane equation and solves for one coordinate	dM1
		Obtains the other 2 coordinates	dM1
	$(2, 3, 1)$	Correct coordinates Accept if written as a column vector but not with **i, j, k**	A1
			(4)
(b)	$\mathbf{PQ}=\begin{pmatrix}2+4\\3+5\\1-3\end{pmatrix}, \mathbf{PR}=\begin{pmatrix}-1+4\\6+5\\4-3\end{pmatrix}, \mathbf{RQ}=\begin{pmatrix}2+1\\3-6\\1-4\end{pmatrix}$	Attempts 2 vectors in plane PQR (Must use the given coordinates of P, R and their coordinates of Q	M1
	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\6 & 8 & -2\\3 & 11 & 1\end{vmatrix}=\begin{pmatrix}30\\-12\\42\end{pmatrix}$	Attempt vector product between 2 vectors in PQR. **Depends on the first mark.**	dM1
	$\begin{pmatrix}5\\-2\\7\end{pmatrix}\bullet\begin{pmatrix}2\\3\\1\end{pmatrix}=11$	Uses any of P, Q or R to find constant. **Depends on the previous mark.**	dM1
	$5x-2y+7z=11$	Any correct Cartesian equation	A1
			(4)

Way 2	$-4a-5b-3c=1$ $2a+3b+c=1$ $-a+6b+4c=1$	Uses the Cartesian form of the equation of a plane, $ax+by+cz=1$, and substitutes the coordinates of each of the 3 points	M1
	Solves to get a value for any of a, b or c		dM1
	Obtains values for the other 2		dM1
	$\frac{5}{11}x-\frac{2}{11}y+\frac{7}{11}z=1$	Any correct Cartesian equation	A1
			(4)

(c)	Reflection of P in Π is $\begin{pmatrix}-4\\-5\\3\end{pmatrix}+2\times"2"\begin{pmatrix}3\\4\\-1\end{pmatrix}\left(=\begin{pmatrix}8\\11\\-1\end{pmatrix}\right)$	Correct strategy for another point on l_3	M1
	$\begin{pmatrix}8\\11\\-1\end{pmatrix}-\begin{pmatrix}-1\\6\\4\end{pmatrix}\left(=\begin{pmatrix}9\\5\\-5\end{pmatrix}\right)$	Attempts direction of l_3. **Depends on the first mark.**	dM1
	$\mathbf{r}=\begin{pmatrix}-1\\6\\4\end{pmatrix}+\lambda\begin{pmatrix}9\\5\\-5\end{pmatrix}$	Forms the equation of l_3 using R (or their reflected P) and their direction. **Depends on the previous mark.**	ddM1
		Any correct equation in vector form	A1 **(4)**
			Total 12

Question Number	Scheme	Notes	Marks
9	$\frac{x^2}{9}+\frac{y^2}{4}=1, \quad y=kx-3$		
(a)	$\frac{x^2}{9}+\frac{(kx-3)^2}{4}=1 \left(\text{or } \frac{x^2}{9}+\frac{k^2x^2-6kx+9}{4}=1\right) \Rightarrow 4x^2+9\left(k^2x^2-6kx+9\right)=36$ Substitutes to obtain a quadratic in x and eliminates fractions		M1
	$\left(9k^2+4\right)x^2-54kx+45=0$*	Correct proof with no errors	A1*
			(2)
(b)	$x=\frac{1}{2}\left(\frac{54k}{9k^2+4}\right)=\frac{27k}{9k^2+4}$ OR $x=\frac{54k\pm\sqrt{\text{discriminant}}}{2\left(9k^2+4\right)}$	Uses ½ sum of roots for the x coordinate OR Solve the equation (by formula), add the 2 roots and halve the result. Must reach x_m. Allow errors in the discriminant	M1
	$y=k\left(\frac{27k}{9k^2+4}\right)-3$ $y=\frac{27k^2-27k^2-12}{9k^2+4}=-\frac{12}{9k^2+4}$	Uses the straight line equation to find y as a single fraction, can be unsimplified Depends on first M mark of (b)	dM1
	$x=\frac{27k}{9k^2+4}, \quad y=-\frac{12}{9k^2+4}$	Fully correct work	A1
			(3)
(c)	$x^2=\frac{729k^2}{\left(9k^2+4\right)^2} \Rightarrow x^2+py^2=\frac{729k^2+144p}{\left(9k^2+4\right)^2}$ Obtains an expression for x^2+py^2 using their coordinates obtained in (b) and obtains a common denominator		M1
	$\frac{729k^2+144p}{\left(9k^2+4\right)^2}=-\frac{12q}{\left(9k^2+4\right)} \Rightarrow 729k^2+144p=-12q\left(9k^2+4\right)$ $729k^2+144p=81\left(9k^2+\frac{16}{9}p\right)$ $\Rightarrow \frac{16}{9}p=4 \Rightarrow p=...$ Correct strategy to obtain a value for p or for q Depends on the first M mark of (c)		dM1
	$p=\frac{9}{4}$ or $q=-\frac{27}{4}$ oe	Correct value (or for q if found first)	A1
	$-12q=81 \Rightarrow q=...$	Correct strategy to obtain a value for the second variable Depends on both previous M marks	ddM1
	$\Rightarrow x^2+\frac{9}{4}y^2=-\frac{27}{4}y$ $p=\frac{9}{4}$ and $q=-\frac{27}{4}$ oe	Both values correct – can be embedded in the equation	A1
			(5)

(c) Way 2	$x=\frac{27k}{9k^2+4},\quad y=-\frac{12}{9k^2+4}\Rightarrow\frac{x}{y}=-\frac{27k}{12}\Rightarrow k=-\frac{4x}{9y}$ Obtains k in terms of x and y using their coordinates found in (b)		M1
	$k=-\frac{4x}{9y}\Rightarrow y=-\frac{12}{9\left(\frac{16x^2}{81y^2}\right)+4}$ or $x=\frac{27\left(-\frac{4x}{9y}\right)}{9\left(\frac{16x^2}{81y^2}\right)+4}$ dM1:Substitutes k into y or x to obtain a Cartesian equation A1: Any correct Cartesian equation Depends on the first M mark of (c)		dM1A1
	$\Rightarrow x^2+\frac{9}{4}y^2=-\frac{27}{4}y$	Rearranges to the form required Depends on both previous M marks of (c)	ddM1
		Correct equation or correct values stated	A1
			Total 10

Question Number	Scheme	Notes	Marks
1(a)	$\frac{dy}{dx}=3\arcsin 2x+3x\frac{1}{\sqrt{1-(2x)^2}}\times 2$ $\left(=3\arcsin 2x+\frac{6x}{\sqrt{1-4x^2}}\right)$	M1: Obtains $p\arcsin qx+\frac{rx}{\sqrt{1-(sx)^2}}$ **or** $p\arcsin qx+\frac{rx}{\sqrt{1-tx^2}}$ $p,\ q,\ r,\ s,\ t>0$ A1: Correct derivative. Allow unsimplified and isw. Allow $\sin^{-1}$ and condone "arsin" but "arsinh" or "arcsinh" is M0	M1 A1
(b)	$x=\frac{1}{4}\Rightarrow\frac{dy}{dx}=\frac{\pi}{2}+\sqrt{3}$	$\frac{\pi}{2}+\sqrt{3}$ only but allow $\frac{1}{2}\pi$ or 0.5π. Terms as a sum in either order. Allow $a=\frac{1}{2},\ b=\sqrt{3}$ Isw following a correct answer.	B1**dep**
	This is a "Hence" question so this mark can only be awarded following full marks in part (a)		
			Total 3

Question Number	Scheme	Notes	Marks
2(a)	$x=-\frac{4}{3}$	$x=-\frac{4}{3}$ or any equivalent **equation**. Allow $x=\pm\frac{4}{3}$	B1
			(1)
(b)(i) **Way 1**	$\frac{a}{e}=\frac{4}{3}$ $b^2=a^2(e^2-1)\Rightarrow 5=a^2\left(\frac{9a^2}{16}-1\right)$	Uses $\frac{a}{e}=\pm\frac{4}{3}$ oe and a correct eccentricity formula and obtains an equation in a. Condone replacing b^2 with 25 if equation is otherwise correct	M1
	$9a^4-16a^2-80=0$ $\Rightarrow(9a^2+20)(a^2-4)=0\Rightarrow a^2=\ldots$	Solves a 3TQ in a^2 (or equation that would lead to a 3TQ) to find a positive real root (usual rules – but if no working seen they must obtain one positive real value of a^2 or a correct to 3 sf which is consistent with their equation). Do not award if confusion with variable e.g., "$(9a^2+20)(a^2-4)=0\Rightarrow a=4$" **Requires previous M mark.**	**d**M1
	$a=2$	Not $a=\pm2$ unless negative rejected	A1
			(3)
Way 2	$\frac{a}{e}=\frac{4}{3}$ $b^2=a^2(e^2-1)\Rightarrow 5=\left(\frac{4e}{3}\right)^2(e^2-1)$	Uses $\frac{a}{e}=\pm\frac{4}{3}$ oe and a correct eccentricity formula and obtains an equation in e. Condone replacing b^2 with 25 if equation is otherwise correct	M1
	$16e^4-16e^2-45=0$ $\Rightarrow(4e^2-9)(4e^2+5)=0\Rightarrow e^2=\ldots$	Solves a 3TQ in e^2 (or equation that would lead to a 3TQ) to find a positive real root (usual rules – but if no working seen they must obtain one positive real value of e^2 or e correct to 3 sf which is consistent with their equation). Do not award if confusion with variable e.g., "$(4e^2-9)(4e^2+5)=0\Rightarrow e=\frac{9}{4}$" **Requires previous M mark.**	**d**M1
	$\left(e=\frac{3}{2}\Rightarrow\right)$ $a=2$	Not $a=\pm2$ unless negative rejected but condone sight of "$e=\pm\frac{3}{2}$" or "$e=-\frac{3}{2}$"	A1
			(3)

Question Number	Scheme	Notes	Marks
2(b)(ii)	$e=\frac{3}{2}\Rightarrow ae=\frac{3}{2}\times 2$ or $ae=\frac{3a^2}{4}=\frac{3}{4}\times 4$ or $ae=c=\sqrt{a^2+b^2}=\sqrt{2^2+5}$	Uses a correct method to obtain a numerical expression for *ae* oe with their values of *a*, *e*, a^2, b^2 etc. however obtained. Condone use of a negative *e* or *a*	M1
	Foci are $(\pm 3, 0)$	Both correct foci as coordinates	A1
	Allow $"\frac{a}{e}=\frac{4}{3}\Rightarrow a=4,\ e=3"$ to access the last M mark only in (b) for $(\pm 12, 0)$ provided the values of both *a* and *e* are clearly seen beforehand		**(2)**
			Total 6
	Note that it is possible to answer (ii) before (i) – e.g., Let foci be $(\pm c,\ 0)$ $a^2e^2=c^2=b^2+a^2=5+a^2$ and $\frac{a}{e}=\frac{a^2}{ae}=\frac{a^2}{c}=\frac{4}{3}\Rightarrow a^2=\frac{4}{3}c$ $\Rightarrow c^2=5+\frac{4}{3}c$ ((i) M1: Uses correct formulae to form an equation in *c* – condone b^2 replaced with 25 as with main scheme) $\Rightarrow 3c^2-4c-15=0\Rightarrow(3c+5)(c-3)=0\Rightarrow c=3$ ((i) **d**M1: Solves 3TQ to find positive real root) $\Rightarrow(\pm 3,\ 0)$ ((i)A1: Correct foci as coordinates) $a=\sqrt{\frac{4}{3}\times 3}$ ((ii) M1: Correct method for *a*) $a=2$ ((ii) A1: Correct value)		

Question Number	Scheme	Notes	Marks
3 **Way 1** **Converts to sinh and cosh**	$4\tanh x - \operatorname{sech} x = 1$ $4\dfrac{\sinh x}{\cosh x} - \dfrac{1}{\cosh x} = 1$ $4\sinh x - 1 - \cosh x = 0$ $4\dfrac{e^x - e^{-x}}{2} - 1 - \dfrac{e^x + e^{-x}}{2} = 0$	Replaces **one** hyperbolic function with its correct exponential equivalent. Allow for correct replacement of just e.g., $\sinh x$ after using $\tanh x = \dfrac{\sinh x}{\cosh x}$ May follow errors but do not allow any further marks if the original equation was reduced to one in a single hyperbolic function.	M1
	$3e^{2x} - 2e^x - 5 = 0$	M1: Obtains an equation which if terms are collected is a 3TQ (or 2TQ with no constant) in e^x A1: Correct 3TQ	M1 A1
	$e^x = \dfrac{2 \pm \sqrt{4+60}}{6} \left(\Rightarrow \dfrac{2+8}{6} = \dfrac{5}{3} \right)$	M1: Solves 3TQ (or 2TQ with no constant) in e^x. Apply usual rules. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. If 2TQ must get a correct non-zero root of their equation. A1: Any correct unsimplified expression for e^x that includes the positive root. Must be exact	M1 A1
	$x = \ln\dfrac{5}{3}$	$\ln\dfrac{5}{3}$, $\ln 1\dfrac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k = \ldots$ No unrejected extra solutions	A1
			Total 6
Way 2 **Straight to e^x**	$4\dfrac{e^x - e^{-x}}{e^x + e^{-x}} - \dfrac{2}{e^x + e^{-x}} = 1$	Replaces **one** hyperbolic function with its correct exponential equivalent	M1
	$3e^{2x} - 2e^x - 5 = 0$	M1: Obtains an equation which if terms are collected is a 3TQ (or 2TQ with no constant) in e^x A1: Correct 3TQ	M1 A1
	$e^x = \dfrac{2 \pm \sqrt{4+60}}{6} \left(\Rightarrow \dfrac{2+8}{6} = \dfrac{5}{3} \right)$	M1: Solves 3TQ (or 2TQ with no constant) in e^x. Apply usual rules. If no working seen they must achieve one correct root of their equation to 3sf which may be complex. If 2TQ must get a correct non-zero root of their equation. A1: Any correct unsimplified expression for e^x that includes the positive root. Must be exact	M1 A1
	$x = \ln\dfrac{5}{3}$	$\ln\dfrac{5}{3}$, $\ln 1\dfrac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k = \ldots$ No unrejected extra solutions	A1
			Total 6
	In Ways 1 & 2, if they form an equation which is not a quadratic in e^x they must achieve the correct exact root of $\dfrac{5}{3}$ to access the middle four marks		

Question Number	Scheme	Notes	Marks
3 **Way 3a** **Squaring (sinh)**	$4\sinh x - 1 = \cosh x$ $16\sinh^2 x - 8\sinh x + 1 = \cosh^2 x$ $16\sinh^2 x - 8\sinh x + 1 = 1 + \sinh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in sinh x	M1
	$15\sinh^2 x - 8\sinh x = 0$	M1: Obtains a 2TQ with no constant or 3TQ in sinh x A1: Correct 2TQ	M1 A1
	$\sinh x = \frac{8}{15}$	Solves 2TQ (with no constant) or 3TQ in sinh x. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \text{arsinh}\frac{8}{15} = \ln\left(\frac{8}{15} + \sqrt{\left(\frac{8}{15}\right)^2 + 1}\right)$ or $15e^{2x} - 16e^x - 15 = 0 \Rightarrow$ $e^x = \frac{16 \pm \sqrt{256 + 900}}{30}$	A correct unsimplified expression for x as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln\frac{5}{3}$	$\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k = ...$ No unrejected extra solutions	A1
			Total 6
Way 3b **Squaring (sech)**	$4\tanh x = 1 + \text{sech}\, x$ $16\tanh^2 x = 1 + 2\,\text{sech}\, x + \text{sech}^2 x$ $16(1 - \text{sech}^2 x) = 1 + 2\,\text{sech}\, x + \text{sech}^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in sech x	M1
	$17\,\text{sech}^2 x + 2\,\text{sech}\, x - 15 = 0$	M1: Obtains a 2TQ (with no constant) or 3TQ in sech x A1: Correct 3TQ	M1 A1
	$(17\,\text{sech}\, x - 15)(\text{sech}\, x + 1) = 0$ $\text{sech}\, x = \frac{15}{17}$	Solves 2TQ with no constant or 3TQ in sech x. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \text{arcosh}\frac{17}{15} = \ln\left(\frac{17}{15} + \sqrt{\left(\frac{17}{15}\right)^2 - 1}\right)$ or $15e^{2x} - 34e^x + 15 = 0 \Rightarrow$ $e^x = \frac{34 \pm \sqrt{1156 - 900}}{30}$	A correct unsimplified expression for x as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln\frac{5}{3}$	$\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k = ...$ No unrejected extra solutions	A1
			Total 6

Question Number	Scheme	Notes	Marks
3 **Way 3c** **Squaring (tanh)**	$4\tanh x - 1 = \text{sech}\, x$ $16\tanh^2 x - 8\tanh x + 1 = \text{sech}^2 x$ $16\tanh^2 x - 8\tanh x + 1 = 1 - \tanh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in tanh x	M1
	$17\tanh^2 x - 8\tanh x = 0$	M1: Obtains a 2TQ with no constant or 3TQ in tanh x A1: Correct 2TQ	M1 A1
	$\tanh x = \frac{8}{17}$	Solves 2TQ with no constant or 3TQ in tanh x. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \text{artanh}\frac{8}{17} = \frac{1}{2}\ln\left(\frac{1+\frac{8}{17}}{1-\frac{8}{17}}\right)$ or $9e^{2x} - 25 = 0 \Rightarrow$ $e^x = \frac{5}{3}$	A correct unsimplified expression for x as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln\frac{5}{3}$	$\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k = ...$ No unrejected extra solutions	A1
			Total 6
Way 3d **Squaring (cosh)**	$4\sinh x = 1 + \cosh x$ $16\sinh^2 x = 1 + 2\cosh x + \cosh^2 x$ $16\cosh^2 x - 16 = 1 + 2\cosh x + \cosh^2 x$	Squares (condone poor squaring) and uses a correct hyperbolic identity to obtain a quadratic equation in cosh x	M1
	$15\cosh^2 x - 2\cosh x - 17 = 0$	M1: Obtains a 2TQ with no constant or 3TQ in cosh x A1: Correct 3TQ	M1 A1
	$(15\cosh x - 17)(\cosh x + 1) = 0$ $\cosh x = \frac{17}{15}$	Solves 2TQ (with no constant) or 3TQ in cosh x. Apply usual rules. If 2TQ must get a correct non-zero root of their equation. If no working seen they must achieve one correct root of their equation to 3sf which may be complex.	M1
	$x = \text{arcosh}\frac{17}{15} = \ln\left(\frac{17}{15} + \sqrt{\left(\frac{17}{15}\right)^2 - 1}\right)$ or $15e^{2x} - 34e^x + 15 = 0 \Rightarrow$ $e^x = \frac{34 \pm \sqrt{1156 - 900}}{30}$	A correct unsimplified expression for x as a ln (or any correct unsimplified expression for e^x if they revert to exponentials). Must be exact	A1
	$x = \ln\frac{5}{3}$	$\ln\frac{5}{3}$, $\ln 1\frac{2}{3}$, $\ln 1.\dot{6}$ only but allow $k = ...$ No unrejected extra solutions	A1
			Total 6

Question Number	Scheme	Notes	Marks
4(a)	$\int \frac{1}{\sqrt{9x^2+16}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{x^2+\frac{16}{9}}}\,dx$ $= \frac{1}{3}\text{arsinh}\left(\frac{3x}{4}\right)$ or $\frac{1}{3}\text{arsinh}\left(\frac{x}{\frac{4}{3}}\right)$ $(+c)$ or $\frac{1}{3}\ln\left(x+\sqrt{x^2+\left(\frac{4}{3}\right)^2}\right)$ $(+c)$	M1: Obtains $p\,\text{arsinh}(qx)$ or $r\ln\left\{x+\sqrt{x^2+s}\right\}$ or $t\ln\left(ux+\sqrt{vx^2+w}\right)$ $p, q, r, s, t, u, v, w > 0$ A1: Any correct expression. Could be unsimplified and isw. The "+c" is not required. Allow $\sinh^{-1}$ and condone "arcsinh". "arcsin" or "arsin" is M0	M1 A1
			(2)
(b)	$\int_{-2}^{2} \frac{1}{\sqrt{9x^2+16}}\,dx$ $= \left[\frac{1}{3}\text{arsinh}\left(\frac{3x}{4}\right)\right]_{-2}^{2}$ or $\left[\frac{2}{3}\text{arsinh}\left(\frac{3x}{4}\right)\right]_{0}^{2}$ $= \frac{1}{3}\text{arsinh}\left(\frac{3\times 2}{4}\right) - \frac{1}{3}\text{arsinh}\left(\frac{3\times -2}{4}\right)$ or $\frac{2}{3}\text{arsinh}\left(\frac{3}{2}\right)$ OR $\left[\frac{1}{3}\ln\left(x+\sqrt{x^2+\frac{16}{9}}\right)\right]_{-2}^{2}$ $= \frac{1}{3}\ln\left(2+\sqrt{2^2+\frac{16}{9}}\right) - \frac{1}{3}\ln\left(-2+\sqrt{(-2)^2+\frac{16}{9}}\right)$ or $\frac{2}{3}\left(\ln\left(2+\sqrt{2^2+\frac{16}{9}}\right) - \ln\left(0+\sqrt{0^2+\frac{16}{9}}\right)\right)$	Substitutes the limits 2 and -2 into an expression of the form $p\,\text{arsinh}(qx)$ or $r\ln\left\{x+\sqrt{x^2+s}\right\}$ or $t\ln\left(ux+\sqrt{vx^2+w}\right)$ $p, q, r, s, t, u, v, w > 0$ and subtracts either way round or obtains an expression for $2[\ldots]_0^{\pm 2}$ The expression does not have to be consistent with their answer to (a). No rounded decimals unless exact values recovered. Any f(0) = 0 can be implied by omission. Condone poor bracketing.	M1
	$\frac{1}{3}\ln\left(\frac{11}{2}+\frac{3\sqrt{13}}{2}\right)$ or $\frac{1}{3}\ln\frac{11+3\sqrt{13}}{2}$ or $\frac{2}{3}\ln\left(\frac{3}{2}+\frac{\sqrt{13}}{2}\right)$ or $\frac{2}{3}\ln\frac{3+\sqrt{13}}{2}$	**d**M1: Obtains an expression of the form $a\ln\left(b+c\sqrt{13}\right)$ *or* $a\ln\left(\frac{d+e\sqrt{13}}{f}\right)$ where a,b,c,d,e,f are exact and >0. Condone poor bracketing. **Requires previous M mark.** A1: Any correct equivalent in an appropriate form (fractions may not be in simplest form) with correct bracketing if necessary and isw. **Must come from correct work**. Allow e.g., $a=\frac{2}{3}$, $b=\frac{3}{2}$, $c=\frac{1}{2}$	**d**M1 A1
	For information the decimal answer is 0.7965038115		**(3)**
			Total 5

Question Number	Scheme	Notes	Marks
5(a)	$\begin{vmatrix} a & a & 1 \\ -a & 4 & 0 \\ 4 & a & 5 \end{vmatrix}$ $= a(4\times5-0)-a(-5a-0)+1(-a^2-(4\times4))$	Uses a correct method for det **A** (implied by two correct parts) to obtain an expression in a	M1
	$\Rightarrow 20a+5a^2-a^2-16=0$ $\Rightarrow a^2+5a-4=0$ $\Rightarrow a=\dfrac{-5+\sqrt{41}}{2}$	Correct exact value oe Condone $\dfrac{-5\pm\sqrt{41}}{2}$	A1
			(2)
(b)(i) **Way 1** $\lvert\mathbf{A}-\lambda\mathbf{I}\rvert$	$\lvert\mathbf{A}-\lambda\mathbf{I}\rvert = \begin{vmatrix} a-\lambda & a & 1 \\ -a & 4-\lambda & 0 \\ 4 & a & 5-\lambda \end{vmatrix}$ $=(a-\lambda)(4-\lambda)(5-\lambda)-a\times-a(5-\lambda)+(-a^2-4(4-\lambda))$ or $\lvert\mathbf{A}-2\mathbf{I}\rvert = \begin{vmatrix} a-2 & a & 1 \\ -a & 2 & 0 \\ 4 & a & 3 \end{vmatrix}$ $=6(a-2)-a\times-3a+(-a^2-8)$	Obtains an expression for $\lvert\mathbf{A}-\lambda\mathbf{I}\rvert$ in terms of a and λ or just a if λ is replaced by 2. If method unclear insist on 2 out of 3 correct parts. May multiply along any row/column. Sarrus leads to the same expressions shown (or the expressions all multiplied by -1 if "=0").	M1
	$\lambda=2\Rightarrow(a-2)\times2\times3+3a^2-a^2-8=0$ $2a^2+6a-20=0 \Rightarrow a^2+3a-10=0$ $\Rightarrow (a-2)(a+5)=0 \Rightarrow a=...$	Following use of $\lambda=2$, forms and solves a 3TQ in a. Apply usual rules. If no working they must obtain one correct solution for their 3TQ which could be complex. Could be implied. **Requires previous M mark.**	**d**M1
	$(a>0 \Rightarrow)a=2$	Correct value of a from correct work. If –5 is offered imply its rejection if 2 alone is used in (ii)	A1
	If a = 2 is arrived at fortuitously, all marks are available for the remainder of the question		**(3)**
(b)(i) **Way 2** $\mathbf{Ax}=2\mathbf{x}$	$\mathbf{Ax}=2\mathbf{x}\Rightarrow$ $ax+ay+z=2x$ $-ax+4y=2y$ $4x+ay+5z=2z$	Uses $\mathbf{Ax}=2\mathbf{x}$ $[or\,(\mathbf{A}-2\mathbf{I})\mathbf{x}=0]$ to obtain three simultaneous equations. Allow if given as two equal vectors.	M1
	$\Rightarrow a^2+3a-10=0$ $\Rightarrow (a-2)(a+5)=0 \Rightarrow a=...$	Forms and solves a 3TQ in a. Apply usual rules. If calculator used must obtain one correct solution for their 3TQ which could be complex. Could be implied. **Requires previous M mark.**	**d**M1
	$(a>0 \Rightarrow)a=2$	Correct value of a from correct work. If –5 is offered imply its rejection if 2 alone is used in (ii)	A1
	If a = 2 is arrived at fortuitously, all marks are available for the remainder of the question		**(3)**

Question Number	Scheme	Notes	Marks
5(b)(ii)	$(2-\lambda)(4-\lambda)(5-\lambda)+4(5-\lambda)+(-4-16+4\lambda)=0$ $\Rightarrow(5-\lambda)\left[(2-\lambda)(4-\lambda)+4-4\right]=0$ $\Rightarrow(5-\lambda)(2-\lambda)(4-\lambda)=0\Rightarrow\lambda=...$	Uses their value of a in a recognisable attempt at a characteristic equation and achieves a real non-zero eigenvalue $\neq 2$. There must be some algebra but it may be poor.	M1
	4 and 5	Both correct (no extra) and from correct work	A1
	For information the cubic is $\pm\left(\lambda^3-11\lambda^2+38\lambda-40\right)=0$		**(2)**
(c)	$\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\text{"4"}\begin{pmatrix}x\\y\\z\end{pmatrix}$ $or\left(\mathbf{A}-\text{"4"}\mathbf{I}\right)\begin{pmatrix}x\\y\\z\end{pmatrix}=0\Rightarrow$ $\begin{matrix}2x+2y+z=4x\\-2x+4y=4y\\4x+2y+5z=4z\end{matrix}$ or $\begin{matrix}-2x+2y+z=0\\-2x=0\\4x+2y+z=0\end{matrix}$ OR $\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\text{"5"}\begin{pmatrix}x\\y\\z\end{pmatrix}$ $or\left(\mathbf{A}-\text{"5"}\mathbf{I}\right)\begin{pmatrix}x\\y\\z\end{pmatrix}=0\Rightarrow$ $\begin{matrix}2x+2y+z=5x\\-2x+4y=5y\\4x+2y+5z=5z\end{matrix}$ or $\begin{matrix}-3x+2y+z=0\\-2x-y=0\\4x+2y+z=0\end{matrix}$ Uses $\mathbf{Ax}=\lambda\mathbf{x}$ or $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$ with their value of a and a real non-zero value of $\lambda\neq 2$ to obtain three simultaneous equations (allow if given as two equal vectors) Alternatively attempts vector product of two rows of $\mathbf{A}-\text{"4"}\mathbf{I}$		M1
	$\pm\begin{pmatrix}0\\-1\\2\end{pmatrix}$ **or** $\pm\begin{pmatrix}1\\-2\\7\end{pmatrix}$	One correct eigenvector. As shown or multiple or with components multiplied by e.g. "k" Accept e.g., $x=0,\ y=-1,\ z=2$	A1
	$\pm\begin{pmatrix}0\\-1\\2\end{pmatrix}$ **and** $\pm\begin{pmatrix}1\\-2\\7\end{pmatrix}$	Both correct eigenvectors. As shown or multiple or with components multiplied by e.g. k Accept $x=...,\ y=...,\ z=...$ Both these 2 A marks could be implied by their normalised eigenvectors	A1
	$\pm\frac{1}{\sqrt{5}}\begin{pmatrix}0\\-1\\2\end{pmatrix},\ \pm\frac{1}{\sqrt{54}}\begin{pmatrix}1\\-2\\7\end{pmatrix}$ oe	M1: A correct method to normalise at least one of their eigenvectors A1: Both correct. Allow any exact equivalents. Isw	M1 A1
	All marks available regardless of how $a=2,\ \lambda_2=4\ \&\ \lambda_3=5$ have been obtained		**(5)**
			Total 12

Question Number	Scheme	Notes	Marks
6(a)	$\frac{dx}{d\theta}=\begin{cases} a(1-\cos\theta) \\ \text{or} \\ a-a\cos\theta \end{cases}$ **or** $\frac{dy}{d\theta}=a\sin\theta$	At least **one** correct derivative	B1
	$a^2(1-\cos\theta)^2+(a\sin\theta)^2$ $=a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)$ $=2a^2(1-\cos\theta)$	Squares and adds their derivatives and uses $\cos^2\theta+\sin^2\theta=1$ to obtain an expression in $\cos\theta$ only (not $\cos^2\theta$) Could be implied	M1
	$=2a^2\left(1-\left(1-2\sin^2\left(\frac{\theta}{2}\right)\right)\right)=4a^2\sin^2\frac{\theta}{2}$	**d**M1: Replaces $\cos\theta$ with $\pm1\ \pm2\sin^2\frac{\theta}{2}$ or equivalent trig work (sign errors only on identities) to obtain an expression in $\sin^2\frac{\theta}{2}$ only **Requires previous M mark.** Can be implied. A1: Achieves $4a^2\sin^2\frac{\theta}{2}$ or $k=4$ from correct work	**d**M1 A1
			(4)
(b)	$\text{S.A.}=(2\pi)\int y\sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}}d\theta$ $=(2\pi)\int_{(0)}^{(2\pi)}a(1-\cos\theta)\left(2a\sin\frac{\theta}{2}\right)d\theta$	Applies $y\sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right\}}$ with their $ka^2\sin^2\frac{\theta}{2}$ and square roots. The result of the square root may be incorrect but must be of the form $p\sin\frac{\theta}{2}$ Allow a slip replacing y but they must not have used x, $\frac{dx}{d\theta}$ or $\frac{dy}{d\theta}$ for y Allow the letter k or an invented value. 2π may be absent or wrong. Integral not required.	M1
	$=(2\pi)2a^2\int_{(0)}^{(2\pi)}\left(\sin\frac{\theta}{2}-\sin\frac{\theta}{2}\cos\theta\right)d\theta$ $\Rightarrow(2\pi)2a^2\int_{(0)}^{(2\pi)}\left(\sin\frac{\theta}{2}-\sin\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)\right)d\theta$ or e.g., $\Rightarrow(2\pi)2a^2\int_{(0)}^{(2\pi)}2\sin^3\frac{\theta}{2}\,d\theta$	Uses trig identity/identities (condoning sign errors) to obtain an expression with arguments of $\frac{\theta}{2}$ only. Allow the letter k or an invented value. 2π may be absent or wrong. Integral not required. **Dependent on previous M mark.**	**d**M1
	Scheme continues...		

Question Number	Scheme	Notes	Marks
6(b) cont.	$\left(=(2\pi)4a^2\int_{(0)}^{(2\pi)}\left(\sin\frac{\theta}{2}-\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)\mathrm{d}\theta\right)$ $S=8\pi a^2\left[-2\cos\frac{\theta}{2}+\frac{2}{3}\cos^3\frac{\theta}{2}\right]_{(0)}^{(2\pi)}$ or e.g., $\pi a^2\left[-16\cos\frac{\theta}{2}+\frac{16}{3}\cos^3\frac{\theta}{2}\right]_{(0)}^{(2\pi)}$	A correct expression for the surface area ignoring limits ft their numerical k, i.e., $S=2k\pi a^2\left[-2\cos\frac{\theta}{2}+\frac{2}{3}\cos^3\frac{\theta}{2}\right]_{(0)}^{(2\pi)}$ oe If they integrate in a piecemeal fashion, award this mark if they have a correct expression for their k when integration is completed – any partial evaluations must be correct for their k	A1ft
	$=8\pi a^2\left[\left(-2\cos\frac{2\pi}{2}+\frac{2}{3}\cos^3\frac{2\pi}{2}\right)-\left(-2\cos 0+\frac{2}{3}\cos^3 0\right)\right]$	Substitutes correct limits and attempts to subtract either way round following a completed attempt at integration with a numerical k. **Requires previous M marks** and must have used 2π . Look for evidence of correct limit substitution and subtraction. There may be slips but insist on limits being applied on all integrations if they have been carried out separately. Algebraic results of integration must be seen	**dd**M1
	$=\frac{64}{3}\pi a^2$	Correct exact answer. Accept equivalent fractions.	A1
	All marks available regardless of how $k=4$ was obtained		**(5)**
			Total 9
	Other integration methods: Allow the second M mark to be available before any attempt at integration is made. Otherwise the second M is only awarded if they **complete** integration without any loss of the required forms (i.e., sign and coefficient errors only and just sign errors only with any trig identities). The first A (ft) mark is for a fully correct expression ignoring limits for their k. The last two marks are the same as the main scheme. For information: Applying parts to $\int\sin\frac{\theta}{2}\cos\theta\ \mathrm{d}\theta$ gives $\frac{2}{3}\left(\cos\frac{\theta}{2}+2\sin\frac{\theta}{2}\right)$ Using addition formulae: $\int\sin\frac{\theta}{2}\cos\theta\ \mathrm{d}\theta=\frac{1}{2}\int\left(\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}\right)\mathrm{d}\theta=\frac{1}{2}\left(2\cos\frac{\theta}{2}-\frac{2}{3}\cos\frac{3\theta}{2}\right)$		

Question Number	Scheme	Notes	Marks
7(a)	$\begin{pmatrix}0\\3\\-2\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}=8\mathbf{i}-2\mathbf{j}-3\mathbf{k}$	M1: Attempts vector product of two vectors in the plane. Unless there is a full clear method they must achieve two correct components A1: $\pm(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})$ or multiple	M1 A1
	Allow any vector notation throughout this question		**(2)**
(b)	l has direction vector $\pm(2\mathbf{j}+\ 2\mathbf{k})$	Correct direction for l	B1
	$(\cos\alpha\ or\ \sin\theta=)$ $\left\lvert\dfrac{"(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})"."(2\mathbf{j}+2\mathbf{k})"}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{2^2+2^2}"}\right\rvert=\left\lvert\dfrac{"(8)(0)+(-2)(2)+(-3)(2)"}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{0^2+2^2+2^2}"}\right\rvert\left(=\left\lvert\dfrac{-10}{\sqrt{77}\times\sqrt{8}}\right\rvert\ or\ \left\lvert\dfrac{-5\sqrt{154}}{154}\right\rvert\right)$ M1: For the scalar product of their normal and direction vector divided by the product of the magnitudes of their vectors. The first expression above oe is sufficient. There must have been a valid attempt at both vectors. Allow copying errors/slips if intention is clear. Modulus not required. A1ft: A correct ft numerical expression with scalar product calculated as shown by second expression or better. Allow a decimal correct to 2sf. Modulus not required. Ignore labelling. Actual decimal is 0.40291148... Implied by awrt 24 or 66 or 114 provided some work and nothing incorrect seen. Allow awrt 0.41, 1.16 or 1.99 if working in radians.		M1 A1ft
	Acute angle between l and P $=90-\alpha=90-66.23968409...$ or $\theta=23.76031591...\Rightarrow 24°$ to the nearest degree	awrt 24 from correct work which could be minimal. Degrees symbol not required. **Mark final answer.**	A1
			(4)
	Note that a vector product could be used: M1: $\left\lvert\dfrac{\lvert"(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})"\times"(2\mathbf{j}+2\mathbf{k})"\rvert}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{2^2+2^2}"}\right\rvert$ A1: $\left\lvert\dfrac{"\sqrt{2^2+16^2+16^2}"}{"\sqrt{8^2+2^2+3^2}"\times"\sqrt{2^2+2^2}"}\right\rvert$ $\left(=\dfrac{2\sqrt{129}}{\sqrt{77}\sqrt{8}}=0.9152389511...\right)$ The modulus of the numerator is required for any marks		
(c) **Way 1** **Parallel planes**	$(\mathbf{i}+2\mathbf{j}+3\mathbf{k}).("8\mathbf{i}-2\mathbf{j}-3\mathbf{k}")=-5$ or $(6\mathbf{i}-3\mathbf{j}-6\mathbf{k}).("8\mathbf{i}-2\mathbf{j}-3\mathbf{k}")=72$	M1: Finds a value for the scalar product of a position vector of a point in the plane or the given point and their normal. A1: –5 or 72 (or 5 or -72 if normal is in the opposite direction). May be seen as a distance e.g., $\dfrac{-5}{\sqrt{"77"}}$	M1 A1
	Shortest distance is $\left\lvert\dfrac{-5-72}{\sqrt{77}}\right\rvert=\dfrac{77}{\sqrt{77}}$ or $\sqrt{77}$	**d**M1: Having attempted both scalar products, obtains a numerical expression for the distance. Award for $\dfrac{\pm"5"\pm"72"}{\sqrt{"8"^2+"2"^2+"3"^2}}$ **Dependent on previous M mark.** A1: Correct exact distance. Isw	**d**M1 A1
			(4)

Question Number	Scheme	Notes	Marks
7(c) **Way 2** **Perp. distance formula**	$(\mathbf{i}+2\mathbf{j}+3\mathbf{k}).(\text{"}8\mathbf{i}-2\mathbf{j}-3\mathbf{k}\text{"})=-5$	M1: Finds a value for the scalar product of a position vector to a point the plane and their normal. A1: -5 (or 5 if normal is in the opposite direction)	M1 A1
	"$8x-2y-3z+5=0$" Shortest distance is $\dfrac{\lvert(\text{"}8\text{"})(6)+(\text{"}-2\text{"})(-3)+(\text{"}-3\text{"})(-6)+\text{"}5\text{"}\rvert}{\sqrt{\text{"}8\text{"}^2+\text{"}2\text{"}^2+\text{"}3\text{"}^2}}$ $=\dfrac{77}{\sqrt{77}}$ or $\sqrt{77}$	**d**M1: Uses distance formula with their normal and plane equation to reach a numerical expression for the distance. Condone sign slip on their -5 and their *d* must not be zero. **Dependent on previous M mark.** A1: Correct exact distance. Isw	**d**M1 A1
			(4)
Way 3 **Projection /resolving formula**	Let Q be the point on the plane (1, 2, 3) then $\overrightarrow{PQ}=(\mathbf{i}+2\mathbf{j}+3\mathbf{k})-(6\mathbf{i}-3\mathbf{j}-6\mathbf{k})$ $=-5\mathbf{i}+5\mathbf{j}+9\mathbf{k}$	M1: Attempts vector from given point to a point on the plane A1: Correct vector $(\pm)$	M1 A1
	Shortest distance is $\lvert\overrightarrow{PQ}.\mathbf{n}\rvert=$ $\dfrac{\lvert(\text{"}-5\mathbf{i}+5\mathbf{j}+9\mathbf{k}\text{"}).(\text{"}8\mathbf{i}+-2\mathbf{j}+-3\mathbf{k}\text{"})\rvert}{\sqrt{\text{"}8\text{"}^2+\text{"}2\text{"}^2+\text{"}3\text{"}^2}}=...$ $=\dfrac{77}{\sqrt{77}}$ *or* $\sqrt{77}$	**d**M1: Uses formula with their vectors to reach a numerical expression for the distance **Dependent on previous M mark.** A1: Correct exact distance. Isw	**d**M1 A1
			(4)
Way 4 **Example of method involving the point where the line meets plane**	Line through given point in direction of normal is $r=(6\mathbf{i}-3\mathbf{j}-6\mathbf{k})+\lambda(8\mathbf{i}-2\mathbf{j}-3\mathbf{k})$ & meets plane "$8x-2y-3z+5=0$" when $8(6+8\lambda)-2(-3-2\lambda)-3(-6-3\lambda)+5=0$ $\Rightarrow \lambda=-1$	M1: Uses line through given point in the direction of their normal and substitutes into their plane to find a value for λ. The *d* in their plane equation must not be zero A1: Correct value	M1 A1
	$\lvert-1(\text{"}8\mathbf{i}+-2\mathbf{j}+-3\mathbf{k}\text{"})\rvert=\sqrt{\text{"}8\text{"}^2+\text{"}2\text{"}^2+\text{"}3\text{"}^2}$ Or point of intersection is (6 –"8", –3 –"–2", –6 –"–3") = (–2, –1, –3) and distance is $\sqrt{(6-\text{"}-2\text{"})^2+(-3-\text{"}-1\text{"})^2+(-6-\text{"}-3\text{"})^2}$ $\Rightarrow\sqrt{77}$	**d**M1: Attempts $\lvert\lambda\mathbf{n}\rvert$ or finds point on the plane and obtains numerical expression for distance between this point and the given point **Dependent on previous M mark.** A1: Correct exact distance. Isw	**d**M1 A1
			(4)
	Marks are scored through the ay which is the best overall match for the attempt. Credit for work done in (b) is only available for part (c) if it is used in part (c).		
			Total 10

Question Number	Scheme	Notes	Marks
8(a)	$I_n=\int\cos^n x\,\mathrm{d}x=\int\cos x\cos^{n-1}x\,(\mathrm{d}x)$	Correct split. Could be implied by their work	M1
Way 1	$=\sin x\cos^{n-1}x+\int(n-1)\cos^{n-2}x\sin^2 x\,(\mathrm{d}x)$	Obtains $p\sin x\cos^{n-1}x+\int q\cos^{n-2}x\sin^2 x\,(\mathrm{d}x)$ oe **Requires previous M mark.**	**d**M1
	$=\sin x\cos^{n-1}x+\int(n-1)\cos^{n-2}x\left(1-\cos^2 x\right)(\mathrm{d}x)$	Replaces $\sin^2 x$ with $1-\cos^2 x$ to achieve a correct expression for I_n	A1
	$=\sin x\cos^{n-1}x+(n-1)I_{n-2}-(n-1)I_n$ $\Rightarrow I_n=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}I_{n-2}$ *	Proceeds to the given answer with at least one intermediate step and no errors. Condone missing "dx"s but there must be no missing arguments. Any clear bracketing error must be recovered **before** given answer.	A1*
			(4)
Way 2	$I_n=\int\cos^n x\,\mathrm{d}x=\int\cos^2 x\cos^{n-2}x\,(\mathrm{d}x)$ $=\int\left(1-\sin^2 x\right)\cos^{n-2}x\,(\mathrm{d}x)$	Correct split **and** replaces $\cos^2 x$ with $1-\sin^2 x$	M1
	$=\int\left(\cos^{n-2}x-\cos^{n-2}x\sin^2 x\right)(\mathrm{d}x)$ $=\int\cos^{n-2}x\,(\mathrm{d}x)-\int\left(\sin x\sin x\cos^{n-2}x\right)(\mathrm{d}x)=\ldots$ M1: Expands, splits and obtains $p\int\cos^{n-2}x\,(\mathrm{d}x)+q\cos^{n-1}x\sin x+\int r\cos^n x\,(\mathrm{d}x)$ oe **Requires previous M mark.** A1: Correct expression for I_n: $\int\cos^{n-2}x\,(\mathrm{d}x)-\left(-\frac{1}{n-1}\cos^{n-1}x\sin x+\int\frac{1}{n-1}\cos^n x\,(\mathrm{d}x)\right)$ oe		**d**M1 A1
	$=I_{n-2}+\frac{1}{n-1}\cos^{n-1}x\sin x-\frac{1}{n-1}I_n$ $\Rightarrow I_n=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}I_{n-2}$ *	Proceeds to the given answer with at least one intermediate step and no errors. Condone missing "dx"s but there must be no missing arguments. Any bracketing error must be recovered **before** given answer.	A1*
			(4)
(b)	$I_n=\frac{1}{n}\left[\cos^{n-1}x\sin x\right]_0^{\frac{\pi}{2}}+\frac{n-1}{n}I_{n-2}$ or $=\frac{1}{n}(n-1)I_{n-2}$ $I_2=\frac{1}{2}\left[\cos^{2-1}x\sin x\right]_0^{\frac{\pi}{2}}+\frac{2-1}{2}I_0$ or $=\frac{1}{2}I_0$	Uses the RF to obtain an expression for I_n in terms of I_{n-2} or I_2 in terms of I_0 Condone if necessary if limits are absent.	M1
	$I_n=\frac{(n-1)(n-3)\ldots5\times3\times1}{n(n-2)(n-4)\ldots6\times4\times2}I_0$ with dots & at least 3 terms in each product (first 2 & last, or first & last 2)	Correct expression for I_n in terms of I_0 oe following correct work including 2 applications of the reduction formula (which could be embedded) **prior** to this answer. I_0 may have been calculated previously but do not allow just the final printed answer to imply this mark.	A1
	e.g., $I_0=\int_0^{\frac{\pi}{2}}\mathrm{d}x=\frac{\pi}{2}$ or $I_0=[x]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$ or $I_0=\frac{\pi}{2}-0$	Correct value for I_0 - requires written evidence of integration (minimal)	B1
	$\therefore I_n=\frac{(n-1)(n-3)\ldots5\times3\times1}{n(n-2)(n-4)\ldots6\times4\times2}\times\frac{\pi}{2}$ * Allow extra terms in both products.	Proceeds to given answer. Requires all previous marks. Withhold this mark if no $\frac{1}{k}\left[\cos^{k-1}x\sin x\right]_0^{\frac{\pi}{2}}$ is seen or expression just disappears – one such expression must be replaced by "0" or have substitution seen	A1*
	Attempts via proof by induction will be reviewed.		**(4)**
	Attempts may be seen via $I_n=\frac{(n-1)(n-3)\ldots3}{n(n-2)\ldots4}I_2$ and $I_2=\frac{1}{2}\left[\theta+\frac{1}{2}\sin2\theta\right]_0^{\frac{\pi}{2}}=\frac{1}{2}\times\frac{\pi}{2}$		

Question Number	Scheme	Notes	Marks
8(c)	$\int_0^{\frac{\pi}{2}} \cos^6 x \sin^2 x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^6 x\left(1-\cos^2 x\right)\mathrm{d}x$	Replaces $\sin^2 x$ with $1-\cos^2 x$ Can be implied by an attempt at $I_6 - I_8$	M1
	$= I_6 - I_8 = \left(\frac{5\times3\times1}{6\times4\times2} - \frac{7\times5\times3\times1}{8\times6\times4\times2}\right)\frac{\pi}{2}$	Any correct numerical expression for the integral	A1
	$\left(= \frac{5}{32}\pi - \frac{35}{256}\pi = \right)\frac{5}{256}\pi$ oe	Correct exact value. Accept equivalent fractions and allow e.g., $\left(\frac{5}{128}\right)\frac{\pi}{2}$	A1
	This is a "Hence" and requires clear use of $I_6 - I_8$ For the A marks there must be no evidence that the answer has been arrived at without using part (b). There is no credit in (b) for work in (c). Just "$I = \frac{5}{256}\pi$" is 0/3 but just "$I_6 - I_8 = \frac{5}{256}\pi$" is 3/3		
			(3)
			Total 11

Question Number	Scheme	Notes	Marks
9(a)(i)	$b^2 = a^2\left(1-e^2\right) \Rightarrow 1 = 9\left(1-e^2\right)$ $\Rightarrow e^2 = \ldots\left(\frac{8}{9}\right),\ e = \frac{2\sqrt{2}}{3}$ or $\frac{\sqrt{8}}{3}$	M1: Uses a correct eccentricity formula with correct values for a and b and obtains a value for e^2 or e A1: Correct value for e (not $\pm$) Could be implied	M1 A1
	Foci are $\left(\pm 2\sqrt{2}, 0\right)$ or $\left(\pm\sqrt{8}, 0\right)$	B1: Both correct foci as coordinates Condone any use of a negative e **Note that this is not an ft mark.**	B1
			(3)
(a)(ii)	$x = \pm\frac{9\sqrt{2}}{4}$ or $\pm\frac{9\sqrt{8}}{8}$ or $\pm\frac{9}{\sqrt{8}}$ oe Both correct **equations**. Requires single fraction. **Allow ft**: $x = \pm\frac{\mathbf{3}}{\text{their } e}$ computed into a single fraction, condoning e < 0 Allow "$x_1 = \ldots,\ x_2 = \ldots$" Condone, e.g., "$x = \pm\frac{a}{e} = \frac{9\sqrt{2}}{4}$ or $-\frac{9\sqrt{2}}{4}$" but just "$\frac{a}{e} = \pm\frac{9\sqrt{2}}{4}$" is B0		B1ft
			(2)
(b) **Way 1** $PF = ePM$	$\|PF_1\| = e\|PM_1\|$ or $\|PF_2\| = e\|PM_2\|$ oe	States this definition of an ellipse.	M1
	$\|PF_1\|+\|PF_2\| = \underline{e\left(\|PM_1\|+\|PM_2\|\right)}$ or $\underline{e\left(\|M_1M_2\|\right)}$ "$\frac{2\sqrt{2}}{3}$" $\times 2\times$ "$\frac{9\sqrt{2}}{4}$" oe or $\|PF_1\|+\|PF_2\| =$ $=$ "$\frac{2\sqrt{2}}{3}$"$\left(\text{"}\frac{9\sqrt{2}}{4}\text{"} - x\right) +$ "$\frac{2\sqrt{2}}{3}$"$\left(\text{"}\frac{9\sqrt{2}}{4}\text{"} + x\right)$	Correct method for a numerical expression (or with cancelling "x"s) for $\|PF_1\|+\|PF_2\|$ with their e and directrix. **One of the underlined expressions must be seen for the first approach.** **Requires previous M mark.**	**dM1**
	$= 6$ *	Fully correct proof. Modulus signs are not required.	A1*
Way 1 **Guidance**	If they work in a and e, $e \times 2 \times \frac{a}{e}$ is only acceptable if $\underline{e\left(\|PM_1\|+\|PM_2\|\right)}$ or $\underline{e\left(\|M_1M_2\|\right)}$ is seen (as with using the values) and $e\left(\frac{a}{e} - x\right) + e\left(\frac{a}{e} + x\right)\ (\Rightarrow 2a)$ is acceptable but note in both these general cases the second M mark becomes available when $a = 3$ is substituted. **<u>The second M is not available for any work which relies on</u>** $\|PF_1\| = \|PF_2\|$ **<u>Their proof needs to be shown to be valid for any position of *P*</u>** So $\|PF_1\| + \|PF_2\| = \frac{2\sqrt{2}}{3}\times\frac{9\sqrt{2}}{4} + \frac{2\sqrt{2}}{3}\times\frac{9\sqrt{2}}{4}$ or using $e\times\frac{a}{e} + e\times\frac{a}{e}$ cannot score the second M without $\underline{e\left(\|PM_1\|+\|PM_2\|\right)}$ or $\underline{e\left(\|M_1M_2\|\right)}$ being seen. If e appears as a value it must be correct for the final mark. Just $\|PF_1\|+\|PF_2\| = 2a = 2\times 3 = 6$ is 0/3 Having earned the first mark in Way 1, some candidates proceed to work with a specific point on the ellipse as in Way 2. Further credit is only available if they clearly state e.g, "$\|PF_1\|+\|PF_2\|$ is constant for any P"		**(3)**

Question Number	Scheme	Notes	Marks
9(b) **Way 2** $PF_1+PF_2=k$	$\lvert PF_1\rvert+\lvert PF_2\rvert=\lvert QF_1\rvert+\lvert QF_2\rvert$ where P and Q are any points on the ellipse oe	States this oe definition of an ellipse, justified by explanation. Accept e.g., "$\lvert PF_1\rvert+\lvert PF_2\rvert$ is constant for any P"	M1
	e.g. Q is where E crosses positive x-axis $\Rightarrow \lvert PF_1\rvert+\lvert PF_2\rvert = 3-"2\sqrt{2}"+3+"2\sqrt{2}"$ Q is where E crosses positive y-axis $\Rightarrow \lvert PF_1\rvert+\lvert PF_2\rvert = 2\sqrt{1^2+"2\sqrt{2}"^2}$ Q is on E directly above F_1 $\Rightarrow \lvert PF_1\rvert+\lvert PF_2\rvert =$ $\sqrt{1-\frac{("2\sqrt{2}"^2)}{9}}+\sqrt{\left(2\times"2\sqrt{2}"\right)^2+1-\frac{("2\sqrt{2}"^2)}{9}}$	Correct method for a numerical value for $\lvert PF_1\rvert+\lvert PF_2\rvert$ using another point on the ellipse and their foci. **Requires previous M mark.**	dM1
	$=6$ *	Fully correct proof. Modulus signs are not required.	A1*
			(3)
Way 3 **Point in terms of θ**	$P(3\cos\theta,\sin\theta)$ $\lvert PF_1\rvert^2=\left(3\cos\theta-"2\sqrt{2}"\right)^2+\sin^2\theta$ or $\lvert PF_2\rvert^2=\left(3\cos\theta+"2\sqrt{2}"\right)^2+\sin^2\theta$	Correct general point in parametric form and applies Pythagoras for the distance (or its square) to either of their foci. Allow in terms of a, b and θ	M1
	$\lvert PF_1\rvert+\lvert PF_2\rvert=$ $\sqrt{8\cos^2\theta-12\sqrt{2}\cos\theta+9}+\sqrt{8\cos^2\theta+12\sqrt{2}\cos\theta+9}$	Correct method for $\lvert PF_1\rvert+\lvert PF_2\rvert$ with their foci. Two three term quadratic expressions required but allow the second to be implied if its correct square root is seen. Score when a and b are substituted. **Requires previous M mark.**	dM1
	$\lvert PF_1\rvert+\lvert PF_2\rvert=$ $3-2\sqrt{2}\cos\theta+3+2\sqrt{2}\cos\theta=6$*	Fully correct proof. Modulus signs are not required. The intermediate step shown oe is required for this Way.	A1*
			(3)
Way 4 **Point in terms of x**	$P\left(x,\sqrt{1-\frac{x^2}{9}}\right)$ or $P\left(x,\sqrt{\frac{9-x^2}{9}}\right)$ $\lvert PF_1\rvert^2=\left("2\sqrt{2}"-x\right)^2+1-\frac{x^2}{9}$ or $\lvert PF_2\rvert^2=\left(x+"2\sqrt{2}"\right)^2+1-\frac{x^2}{9}$	Correct general point in terms of x and applies Pythagoras for the distance (or its square) to either of their foci. Allow in terms of a, b and x.	M1
	$\lvert PF_1\rvert+\lvert PF_2\rvert=\sqrt{\frac{8}{9}x^2-4\sqrt{2}x+9}+\sqrt{\frac{8}{9}x^2+4\sqrt{2}x+9}$	Correct method for $\lvert PF_1\rvert+\lvert PF_2\rvert$ with their foci. Two three term quadratic expressions required but allow the second to be implied if its correct square root is seen. Score when a and b are substituted. **Requires previous M mark.**	dM1
	$\lvert PF_1\rvert+\lvert PF_2\rvert=3-\frac{2\sqrt{2}}{3}x+3+\frac{2\sqrt{2}}{3}x=6$*	Fully correct proof. Modulus signs are not required. The intermediate step shown oe is required for this Way.	A1*
	Creditworthy alternative approaches will be reviewed		**(3)**

Question Number	Scheme	Notes	Marks
9(c)	$x^2+9(2x+c)^2=9$ or $\frac{x^2}{9}+(2x+c)^2=1$	Substitutes line into the ellipse equation. Condone slips provided intention clear.	M1
	$37x^2+36cx+9c^2-9=0$ or e.g., $\frac{37}{9}x^2+4cx+c^2-1=0$	Correct quadratic with x^2 terms collected (could be implied)	A1
	½ (sum of roots) $\Rightarrow (x=)\frac{-18c}{37}$ or $(x=)\frac{1}{2}\left(\frac{-36c+\sqrt{(36c)^2-4(37)(9c^2-9)}}{2(37)}+\frac{-36c-\sqrt{(36c)^2-4(37)(9c^2-9)}}{2(37)}\right)$ M1: Correct attempt at ½ (sum of roots), i.e., $-\frac{b}{2a}$ for their quadratic. Ignore how the expression is labelled. **Requires previous M mark.** A1: Any correct **equation** in x and c Allow this mark if e.g., x is seen as M_x		**d**M1 A1
	$\Rightarrow c=\text{"}-\frac{37}{18}\text{"}x\Rightarrow y=2x+\left(\text{"}-\frac{37}{18}\text{"}\right)x$ or $x=\text{"}-\frac{18}{37}\text{"}c\Rightarrow y=2\times\text{"}-\frac{18}{37}\text{"}c+c\Rightarrow\ldots\left(y=\frac{c}{37}\Rightarrow\frac{y}{x}=-\frac{1}{18}\right)$	Substitutes their $c=px$ into the line to obtain an equation in ***x* and *y*** only. Allow e.g., x_M and y_M and condone e.g., suffixes of P & Q This may also be achieved by e.g., finding y in terms of c and then eliminating c with their equation in x and c Must not be using "M_x" or "M_y" etc. but imply this mark from a locus equation in x and y or $x_{...}$ and $y_{...}$ with appropriate suffixes **Requires both previous M marks**	**dd**M1
	$\Rightarrow y_{...}=-\frac{1}{18}x_{...}$ oe $\therefore$ l passes through the origin oe *	Obtains correct equation for locus (accept equivalents) **and** makes conclusion e.g., "passes/goes through origin/O/(0,0)" but allow "shown"/"as required"/"QED" etc. **Requires all previous marks.**	A1*
			(6)
			Total 13
			PAPER TOTAL: 75

Question Number	Scheme	Notes	Marks
1	$7\cosh x + 3\sinh x = 2e^x + 7 \Rightarrow$ $7\left(\frac{e^x + e^{-x}}{2}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right) = 2e^x + 7$ $\left\{\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7\right\}$	Substitutes at least one correct exponential form for either of the hyperbolic terms and achieves an **equation** in exponentials and constants alone	M1
	$\Rightarrow 7\left(e^{2x} + 1\right) + 3\left(e^{2x} - 1\right) = 4e^{2x} + 14e^x$ $\left\{\Rightarrow 5e^{2x} + 2 = 2e^{2x} + 7e^x\right\}$	Multiplies through by e^x to obtain any equation that would form a 3TQ in e^x if like terms were collected	M1
	$\Rightarrow 6e^{2x} - 14e^x + 4 = 0 \quad \left\{3e^{2x} - 7e^x + 2 = 0\right\}$	A correct three term quadratic in e^x. Could be implied by a correct root even if terms have not been collected.	A1
	$\Rightarrow \left(3e^x - 1\right)\left(e^x - 2\right) = 0 \Rightarrow e^x = \ldots$	Solves their 3TQ - usual rules. One correct root for their quadratic if no working. Ignore labelling of the roots even if e.g.,"x" is used.	M1
	$x = \ln 2,\ \ln\frac{1}{3}$	Both correct and simplified but do not isw if there are **other answers.** Allow $-\ln\frac{1}{2}$ for $\ln 2$ and $-\ln 3$ or $\ln 3^{-1}$ for $\ln\frac{1}{3}$	A1
	Answer only is 0/5		**Total 5**
	Note that it is possible to multiply through by e^{-x} to form an equation in e^{-2x}, e^{-x} and constants. Score as main scheme, e.g., $\frac{7}{2}e^x + \frac{7}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} = 2e^x + 7$ $\Rightarrow \frac{7}{2} + \frac{7}{2}e^{-2x} + \frac{3}{2} - \frac{3}{2}e^{-2x} = 2 + 7e^{-x}$ (M1) $\Rightarrow 2e^{-2x} - 7e^{-x} + 3 = 0$ (A1) $\left(2e^{-x} - 1\right)\left(e^{-x} - 3\right) = 0 \Rightarrow e^{-x} = \frac{1}{2},\ 3$ (M1) $\Rightarrow e^x = 2,\ \frac{1}{3} \Rightarrow x = \ln 2,\ \ln\frac{1}{3}$ (A1)		

Question Number	Scheme	Notes	Marks
2	Condone poor notation e.g., determinant lines used for matrix bracketing		
(a)	$\det\begin{pmatrix}2&0&0\\0&1&4\\3&-2&-3\end{pmatrix}\ \left\{=2\times(-3+8)\right\}\ =10$	Correct value for determinant, seen or stated and not just in a final answer	B1
	$\left\{\text{Minors}:\begin{pmatrix}5&-12&-3\\0&-6&-4\\0&8&2\end{pmatrix}\Rightarrow\right\}\ \text{Cofactors}:\begin{pmatrix}5&12&-3\\0&-6&4\\0&-8&2\end{pmatrix}$	Attempts the cofactor matrix with at least 6 correct elements	M1
	Inverse is $\dfrac{1}{\text{"10"}}\begin{pmatrix}5&0&0\\12&-6&-8\\-3&4&2\end{pmatrix}$ or e.g., $\begin{pmatrix}\frac{1}{2}&0&0\\\frac{6}{5}&-\frac{3}{5}&-\frac{4}{5}\\-\frac{3}{10}&\frac{2}{5}&\frac{1}{5}\end{pmatrix}$	Correct inverse but allow ft on their "10". Allow equivalent fractions/decimals. A0 if clearly obtained incorrectly	A1ft
	Work to obtain Adj(**M**) must be seen but it may be minimal, e.g., sight of the matrix of minors followed by the correct answer is acceptable. Note that B0 M1 A1 is possible.		**(3)**
(b)	$\dfrac{1}{10}\begin{pmatrix}5&0&0\\12&-6&-8\\-3&4&2\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\ldots$	Multiplies their $\mathbf{M}^{-1}$ by $\begin{pmatrix}u\\v\\w\end{pmatrix}$ Must use a matrix other than **M** – not just changed by application of determinant. Condone sight of $\mathbf{vM}^{-1}=\ldots$ but must not be a clearly incorrect multiplication method	M1
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\dfrac{1}{10}\begin{pmatrix}5u\\12u-6v-8w\\-3u+4v+2w\end{pmatrix}$ or $\begin{pmatrix}\frac{1}{2}u\\\frac{6}{5}u-\frac{3}{5}v-\frac{4}{5}w\\-\frac{3}{10}u+\frac{2}{5}v+\frac{1}{5}w\end{pmatrix}$ or $\dfrac{1}{d}\begin{pmatrix}5u\\12u-6v-8w\\-3u+4v+2w\end{pmatrix}$ or $\begin{pmatrix}\frac{5}{d}u\\\frac{12}{d}u-\frac{6}{d}v-\frac{8}{d}w\\-\frac{3}{d}u+\frac{4}{d}v+\frac{2}{d}w\end{pmatrix}$ A1ft: Two correct vector components, coordinates or equations, ft their $d\neq 0$ A1ft: All three correct ft their non-zero $d\neq 0$ Must be exact (and not rounded decimals for ft) **These ft marks are not available for an incorrect** Adj(**M**)		A1ft A1ft
			(3)
Alt Using **M**	$2x=u$, $y+4z=v$, $3x-2y-3z=w$ $\Rightarrow$ $x=\ldots$, $y=\ldots$, $z=\ldots$	Uses $\mathbf{M}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}u\\v\\w\end{pmatrix}$ and finds x, y and z as functions of u, v and w Condone sight of $\mathbf{vM}=\ldots$ but must not be a clearly incorrect multiplication method	M1
	$x=\frac{1}{2}u$ $y=\frac{6}{5}u-\frac{3}{5}v-\frac{4}{5}w$ $z=-\frac{3}{10}u+\frac{2}{5}v+\frac{1}{5}w$	A1: Two correct equations A1: All three correct Any form with terms collected	A1 A1
			(3)

Question Number	Scheme	Notes	Marks
2(c)	$3x-7y+2z=-3 \Rightarrow 3\left(\frac{1}{2}u\right)-7\left(\frac{6}{5}u-\frac{3}{5}v-\frac{4}{5}w\right)+2\left(-\frac{3}{10}u+\frac{2}{5}v+\frac{1}{5}w\right)=-3$	Substitutes their expressions into the equation for Π_1	M1
	$-15u+10v+12w=-6$	Correct **equation.** Terms in any order but constant isolated. Accept any integer multiples.	A1
			(2)
			Total 8
Alts	To gain any marks by an alternative approach, a complete attempt at a Cartesian equation for Π_2 must be made by a viable strategy e.g.,		
	general point on $3x-7y+2z=-3$ is $\left(s,\ t,\ -\frac{3}{2}s+\frac{7}{2}t-\frac{3}{2}\right)$ $\begin{pmatrix}2&0&0\\0&1&4\\3&-2&-3\end{pmatrix}\begin{pmatrix}s\\t\\-\frac{3}{2}s+\frac{7}{2}t-\frac{3}{2}\end{pmatrix} \Rightarrow \begin{matrix}u=2s\\v=-6s+15t-6\\w=\frac{15}{2}s-\frac{25}{2}t+\frac{9}{2}\end{matrix} \Rightarrow \begin{matrix}v=-3u+15t-6\\t=-\frac{2}{25}\left(w-\frac{15}{2}\left(\frac{u}{2}\right)-\frac{9}{2}\right)\end{matrix}$ $\Rightarrow v=-3u-\frac{6}{5}w+\frac{9}{2}u+\frac{27}{5}-6$ Obtains a plane equation in any Cartesian form		M1
	$\left\{v=\frac{3}{2}u-\frac{6}{5}w-\frac{3}{5}\Rightarrow\right\}$ $-15u+10v+12w=-6$	Correct **equation.** Terms in any order but constant isolated. Accept any integer multiples.	A1
			(2)
			Total 8

Question Number	Scheme	Notes	Marks
3(a) Way 1 Identities first then squares	$y=\frac{1}{2}(\tan x+\cot x)\Rightarrow\frac{dy}{dx}=\frac{1}{2}(\sec^2 x-\text{cosec}^2 x)$ oe	Correct derivative. Any equivalent.	B1
	$=\frac{1}{2}\left(1+\tan^2 x-(1+\cot^2 x)\right)$ $\left\{=\frac{1}{2}(\tan^2 x-\cot^2 x)\right\}$	Applies $\sec^2 x=\pm\tan^2 x\pm1$ and $\text{cosec}^2 x=\pm\cot^2 x\pm1$ to their derivative	M1
	$\left(\frac{dy}{dx}\right)^2=\frac{1}{4}(\tan^4 x+\cot^4 x-2\tan^2 x\cot^2 x)$	Squares to a 3 term expression (or 4 if middle terms uncollected) $2\tan^2 x\cot^2 x$ can be seen as 2 **Requires previous M mark.**	**d**M1
	$\left\{1+\left(\frac{dy}{dx}\right)^2=1+\frac{1}{4}(\tan^4 x+\cot^4 x-2)\right\}$ $\Rightarrow\frac{1}{4}(\tan^4 x+\cot^4 x+2)$ or $\frac{1}{4}\tan^4 x+\frac{1}{4}\cot^4 x+\frac{1}{2}$ **Not implied. Must be seen**	Adds the 1 and achieves either expression shown but allow the constant to be multiplied by $\tan^2 x\cot^2 x$ May be seen as e.g., $\frac{1}{2}\sqrt{\tan^4 x+\cot^4 x+2\tan^2 x\cot^2 x}$	A1
	$s=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(\tan^2 x+\cot^2 x)\,dx$ * Allow $\int\frac{1}{2}(\tan^2 x+\cot^2 x)$ or $\frac{1}{2}\int\tan^2 x+\cot^2 x$	M1: Applies the arc length formula with their $\frac{dy}{dx}$ A1: Correct result achieved with no clear mathematical errors seen. Condone omission of "dx" and/or limits and **occasional** missing arguments.	M1 A1*
	Converting to sin & cos: likely to score max of 100010 unless tan & cot are convincingly recovered		**(6)**
Way 2 Squares first then identities	$y=\frac{1}{2}(\tan x+\cot x)\Rightarrow\frac{dy}{dx}=\frac{1}{2}(\sec^2 x-\text{cosec}^2 x)$ oe	Correct derivative. Any equivalent.	B1
	$\left(\frac{dy}{dx}\right)^2=\frac{1}{4}(\sec^4 x+\text{cosec}^4 x-2\sec^2 x\,\text{cosec}^2 x)$	Squares a derivative **of the correct form** to obtain a 3 (or 4 if middle terms uncollected) term expression.	M1
	$=\frac{1}{4}\left((1+\tan^2 x)^2+(1+\cot^2 x)^2-2(1+\tan^2 x)(1+\cot^2 x)\right)$ $\left\{=\frac{1}{4}(1+2\tan^2 x+\tan^4 x+1+2\cot^2 x+\cot^4 x-2-2\tan^2 x-2\cot^2 x-2\tan^2 x\cot^2 x)\right\}$	Applies $\sec^2 x=\pm\tan^2 x\pm1$ **twice and** $\text{cosec}^2 x=\pm\cot^2 x\pm1$ **twice. Requires previous M mark.**	**d**M1
	$\left\{1+\left(\frac{dy}{dx}\right)^2=1+\frac{1}{4}(\tan^4 x+\cot^4 x-2)\right\}$ $\Rightarrow\frac{1}{4}(\tan^4 x+\cot^4 x+2)$ or $\frac{1}{4}\tan^4 x+\frac{1}{4}\cot^4 x+\frac{1}{2}$ **Not implied. Must be seen**	Adds the 1 and achieves either expression shown but allow the constant to be multiplied by $\tan^2 x\cot^2 x$ May be seen as e.g., $\frac{1}{2}\sqrt{\tan^4 x+\cot^4 x+2\tan^2 x\cot^2 x}$	A1
	$s=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\tan^2 x+\cot^2 x\,dx$ * Allow $\int\frac{1}{2}(\tan^2 x+\cot^2 x)$ or $\frac{1}{2}\int\tan^2 x+\cot^2 x$	M1: Applies the arc length formula with their $\frac{dy}{dx}$ A1: Correct result achieved with no clear mathematical errors seen. Condone omission of "dx" and/or limits and **occasional** missing arguments.	M1 A1*
	Converting to sin & cos: likely to score max of 100010 unless tan & cot are convincingly recovered		**(6)**

Question Number	Scheme	Notes	Marks
3(b)	$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\tan^2 x+\cot^2 x\right)\,dx=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\sec^2 x-1+\text{cosec}^2 x-1\right)\,dx$	Applies $\tan^2 x=\pm\sec^2 x\pm1$ and $\cot^2 x=\pm\text{cosec}^2 x\pm1$ to the integral	M1
	Work in sin and cos must use identities (sign errors only) and lead to a result of the form below after integration condoning the absence of a term in x but allow the last M to be available following a completed attempt at integration.		
	$=\frac{1}{2}\left[\tan x-\cot x-2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$	M1: For $\pm\sec^2 x\to\pm\tan x$ and $\pm\text{cosec}^2 x\to\pm\cot x$ **Requires previous M mark.** A1: Correct integration. Limits not required.	**d**M1 A1
	$\frac{1}{2}\left(\tan\frac{\pi}{3}-\cot\frac{\pi}{3}-\frac{2\pi}{3}-\left(\tan\frac{\pi}{6}-\cot\frac{\pi}{6}-\frac{2\pi}{6}\right)\right)$ $\left\{\frac{1}{2}\left(\sqrt{3}-\frac{2\pi}{3}-\frac{\sqrt{3}}{3}-\left(\frac{\sqrt{3}}{3}-\frac{\pi}{3}-\sqrt{3}\right)\right)\right\}$	Applies the limits (see note below) following any completed attempt at integration. Allow slips provided it is a clear attempt at $\text{f}\left(\frac{\pi}{3}\right)-\text{f}\left(\frac{\pi}{6}\right)$	M1
	Correct answer in any exact simplified form with 2 terms e.g. $\frac{1}{2}\left(\frac{4\sqrt{3}}{3}-\frac{\pi}{3}\right)$, $\frac{2\sqrt{3}}{3}-\frac{\pi}{6}$, $\frac{2}{\sqrt{3}}-\frac{\pi}{6}$, $\frac{1}{3}\left(2\sqrt{3}-\frac{\pi}{2}\right)$, $\frac{4\sqrt{3}-\pi}{6}$		A1
	Note they may apply the limits $\frac{\pi}{4}$ & $\frac{\pi}{6}$ or $\frac{\pi}{3}$ & $\frac{\pi}{4}$ and then double the result.		**(5)**
	Just the answer or decimal answer (0.6311017628) is 0/5		**Total 11**

Question Number	Scheme	Notes	Marks
4	Allow any suitable vector notation throughout this question.		
(a)	$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}=\begin{pmatrix}2\\4\\-5\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}\Rightarrow\ldots$ or $\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}2\\0\\-5\end{pmatrix}=\begin{pmatrix}3\\6\\-2\end{pmatrix}.\begin{pmatrix}2\\0\\-5\end{pmatrix}\Rightarrow\ldots$ $-x+3y+3z=-5$ and $2x-5z=16$	M1: Uses **r.n** = **a.n** at least once to obtain a plane equation A1: Both correct equations. Accept in **r.n** = p form	M1 A1
	e.g., $x=\dfrac{16+5z}{2}$	Obtains one variable (may be written as parameter for all marks) in terms of one of the other variables	M1
	$z=\dfrac{2x-16}{5}\Rightarrow x=5+3y+3\left(\dfrac{2x-16}{5}\right)$ $\Rightarrow 5x=25+15y+6x-48\Rightarrow x=-15y+23$ $\left\{x=-15y+23=\dfrac{16+5z}{2}\right\}$	M1: Obtains the variable/parameter in terms of the third variable (or the two other variables in terms of the parameter) A1: Both correct equations	M1 A1 **(M1 on epen)**
	Alternatively, $y=\dfrac{-x+23}{15}=\dfrac{6-z}{6}$ or $z=\dfrac{2x-16}{5}=6-6y$		
	$\left\{\dfrac{x-0}{1}=\dfrac{y-\frac{23}{15}}{-\frac{1}{15}}=\dfrac{z+\frac{16}{5}}{\frac{2}{5}}\Rightarrow\right\}$ $\mathbf{r}=\begin{pmatrix}0\\\frac{23}{15}\\-\frac{16}{5}\end{pmatrix}+\lambda\begin{pmatrix}1\\-\frac{1}{15}\\\frac{2}{5}\end{pmatrix}$ M1: Attempts vector equation of line but "**r** =" may be missing. **Requires all previous M marks.** Allow numerical slips but it must be a correct method i.e., an attempt at $\Rightarrow\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}\Rightarrow\mathbf{r}=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\lambda\begin{pmatrix}l\\m\\n\end{pmatrix}$ A1: Any correct equation including "**r** ="		**d**M1 A1
	Or $\left\{\dfrac{x-23}{-15}=\dfrac{y-0}{1}=\dfrac{z-6}{-6}\Rightarrow\right\}$ $\mathbf{r}=\begin{pmatrix}23\\0\\6\end{pmatrix}+\lambda\begin{pmatrix}-15\\1\\-6\end{pmatrix}$ or $\left\{\dfrac{x-8}{\frac{5}{2}}=\dfrac{y-1}{-\frac{1}{6}}=\dfrac{z-0}{1}\Rightarrow\right\}$ $\mathbf{r}=\begin{pmatrix}8\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}\frac{5}{2}\\-\frac{1}{6}\\1\end{pmatrix}$		
	Note that the line may be given in $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$ or $\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$ form		**(7)**

Question Number	Scheme	Notes	Marks
4(a) **Alt** **Finds point and vector product of normals**	$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}=\begin{pmatrix}2\\4\\-5\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}\Rightarrow\ldots$ or $\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}2\\0\\-5\end{pmatrix}=\begin{pmatrix}3\\6\\-2\end{pmatrix}.\begin{pmatrix}2\\0\\-5\end{pmatrix}\Rightarrow\ldots$ $-x+3y+3z=-5$ and $2x-5z=16$	M1: Uses $\mathbf{r.n}=\mathbf{a.n}$ at least once to obtain a plane equation A1: Both correct equations Accept in $\mathbf{r.n}=p$ form	M1 A1
	e.g., $x=0\Rightarrow z=-\dfrac{16}{5}$	Sets one variable equal to a value and finds a value for another variable. Correct for their equations if no working.	M1
	$3y=-5-3\left(-\dfrac{16}{5}\right)\Rightarrow y=\dfrac{23}{15}\ \left\{\Rightarrow\left(0,\ \frac{23}{15},\ -\frac{16}{5}\right)\right\}$ Or e.g., $(23,\ 0,\ 6)$, $(8,\ 1,\ 0)$ Points will have the form $(23-15\alpha,\ \alpha,\ 6-6\alpha)$	M1: Proceeds to find a value for the remaining variable. Correct for their equations if no working. A1: Correct values	M1 A1 **(M1 on epen)**
	$\begin{pmatrix}-1\\3\\3\end{pmatrix}\times\begin{pmatrix}2\\0\\-5\end{pmatrix}=\ldots\ \Rightarrow\mathbf{r}=\begin{pmatrix}0\\\frac{23}{15}\\-\frac{16}{5}\end{pmatrix}+\lambda\begin{pmatrix}-15\\1\\-6\end{pmatrix}$ $\left\{\mathbf{r}=\begin{pmatrix}23\\0\\6\end{pmatrix}+\lambda\begin{pmatrix}-15\\1\\-6\end{pmatrix}\quad \mathbf{r}=\begin{pmatrix}8\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-15\\1\\-6\end{pmatrix}\right\}$	**d**M1: Attempts vector product of normals (two correct components if method unclear) and forms vector equation with **point** and direction in correct places but allow for a copying error or mix up with components. **Note** that they could obtain the direction from 2 points on the line. **Requires all previous M marks.** "**r** =" may be missing. A1: Any correct equation including "**r** ="	**d**M1 A1
			(7)

Question Number	Scheme	Notes	Marks
4(b)	Note: If 0/5 allow SC 00010 for a correct volume formula seen **for tetrahedron *ABCD*** e.g., $\frac{1}{6}\left\|\overrightarrow{CD}.\left(\overrightarrow{CA}\times\overrightarrow{CB}\right)\right\|$ Allow with missing modulus but not vector arrows unless implied by further work.		
Way 1 STP inc. $\overrightarrow{CD}$	$\left\|\begin{pmatrix}-15\\1\\-6\end{pmatrix}\right\|=\sqrt{262}\Rightarrow\overrightarrow{CD}=\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}$	Attempts magnitude (allow numerical slip) of their direction vector and scales correctly to length 5	M1
	Let C be the point $(8, 1, 0)$ $\overrightarrow{CA}=\begin{pmatrix}2\\4\\-5\end{pmatrix}-\begin{pmatrix}8\\1\\0\end{pmatrix}=...\left\{\begin{pmatrix}-6\\3\\-5\end{pmatrix}\right\}$ and $\overrightarrow{CB}=\begin{pmatrix}3\\6\\-2\end{pmatrix}-\begin{pmatrix}8\\1\\0\end{pmatrix}=...\left\{\begin{pmatrix}-5\\5\\-2\end{pmatrix}\right\}$	Finds vectors for **any** two edges other than *CD*. Could be implied by a distance calculation **if *C* and/or *D* defined**. This mark is not scored if either vector is in terms of a parameter unless it is assigned a value (or is eliminated appropriately) later.	M1
	$\overrightarrow{CD}.\left(\overrightarrow{CA}\times\overrightarrow{CB}\right)=\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}.\begin{pmatrix}-6\\3\\-5\end{pmatrix}\times\begin{pmatrix}-5\\5\\-2\end{pmatrix}=...\left\{=-\frac{910}{\sqrt{262}}\right\}$	Uses an appropriate scalar triple product with their vectors and finds a value. **Must not include position vectors**. Could be inexact. M0 if clear evidence of an inappropriate method	M1
	$V=\frac{1}{6}\left\|\overrightarrow{CD}.\left(\overrightarrow{CA}\times\overrightarrow{CB}\right)\right\|=...=\frac{455}{3\sqrt{262}}$ or $\frac{455\sqrt{262}}{786}$	**d**M1: Divides their STP result by 6 and obtains a positive value. Could be inexact. Modulus might not be seen. **Requires previous M mark.** A1: A correct **exact** value	**d**M1 A1
			(5)
Way 2 STP not inc. $\overrightarrow{CD}$	$\left\|\begin{pmatrix}-15\\1\\-6\end{pmatrix}\right\|=\sqrt{262}\Rightarrow\overrightarrow{CD}=\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}$	Attempts magnitude (allow numerical slip) of their direction vector and scales correctly to length 5	M1
	Let C be the point $(8, 1, 0)$ $\overrightarrow{AC}=\begin{pmatrix}8\\1\\0\end{pmatrix}-\begin{pmatrix}2\\4\\-5\end{pmatrix}=...\left\{\begin{pmatrix}6\\-3\\5\end{pmatrix}\right\}$ and $\overrightarrow{AB}=\begin{pmatrix}3\\6\\-2\end{pmatrix}-\begin{pmatrix}2\\4\\-5\end{pmatrix}=...\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$	Finds vectors for **any** two edges other than *CD*. Could be implied by a distance calculation **if *C* and/or *D* defined**. (See also comment for second M1 in Way 1 re use of a parameter)	M1
	$\overrightarrow{OD}=\begin{pmatrix}8\\1\\0\end{pmatrix}+\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}\Rightarrow\overrightarrow{AD}=\begin{pmatrix}\frac{-75}{\sqrt{262}}+8\\\frac{5}{\sqrt{262}}+1\\\frac{-30}{\sqrt{262}}\end{pmatrix}-\begin{pmatrix}2\\4\\-5\end{pmatrix}=\begin{pmatrix}\frac{-75}{\sqrt{262}}+6\\\frac{5}{\sqrt{262}}-3\\\frac{-30}{\sqrt{262}}+5\end{pmatrix}$ $\Rightarrow\overrightarrow{AD}.\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=\begin{pmatrix}\frac{-75}{\sqrt{262}}+6\\\frac{5}{\sqrt{262}}-3\\\frac{-30}{\sqrt{262}}+5\end{pmatrix}.\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}6\\-3\\5\end{pmatrix}=...\left\{=-\frac{910}{\sqrt{262}}\right\}$	Uses an appropriate scalar triple product with their vectors and finds a value. **Must not include position vectors**. Could be inexact. M0 if clear evidence of an inappropriate method	M1
	$V=\frac{1}{6}\left\|\overrightarrow{AD}.\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right\|=...=\frac{455}{3\sqrt{262}}$ or $\frac{455\sqrt{262}}{786}$	**d**M1: Divides their STP result by 6 and obtains a positive value. Could be inexact. Modulus might not be seen. **Requires previous M mark.** A1: A correct **exact** value	**d**M1 A1
			(5)

Question Number	Scheme	Notes	Marks
4(b) Way 3 Triangle area + perp. distance to plane & vol. of pyramid	$\left\lvert\begin{pmatrix}-15\\1\\-6\end{pmatrix}\right\rvert=\sqrt{262}\Rightarrow\overrightarrow{CD}=\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}$	Attempts magnitude of their direction vector and scales to length 5. See note after next M below.	M1
	Let C be the point $(8, 1, 0)$ $Area\,\Delta ACD=\frac{1}{2}\left\lvert\overrightarrow{CD}\times\overrightarrow{CA}\right\rvert=\frac{1}{2}\left\lvert\frac{5}{\sqrt{262}}\begin{pmatrix}-15\\1\\-6\end{pmatrix}\times\begin{pmatrix}-6\\3\\-5\end{pmatrix}\right\rvert=\ldots\left\{=\frac{65\sqrt{19}}{2\sqrt{262}}\right\}$ Uses formula to find a value for the area of one of the faces. Must be a full method (vector product and modulus). Condone missing $\frac{1}{2}$ Any attempts by trig/Pythagoras must be complete and credible		M1
	Note: It is possible to obtain the area of a relevant triangle such as ACD by e.g., finding the length of the perpendicular distance of point A to the line and multiplying this by $\frac{1}{2}$ x 5 – in such cases allow the first M for completing a viable attempt at the height of the triangle and the second for the area (Condone missing $\frac{1}{2}$)		
	ΔACD is in Π_1 so perp. height of tetrahedron is shortest dist. of $B(3, 6, -2)$ to $-x+3y+3z=-5$: $\left\lvert\frac{-1\times3+3\times6+3\times(-2)+5}{\sqrt{(-1)^2+3^2+3^2}}\right\rvert=\ldots\left\{\frac{14}{\sqrt{19}}\right\}$	Obtains a value for the perpendicular height via formula or any credible method (examples below)	M1
	Parallel planes : $\begin{pmatrix}3\\6\\-2\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}=9,\ \begin{pmatrix}2\\4\\-5\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}=-5\Rightarrow\left\lvert\frac{-5-9}{\sqrt{(-1)^2+3^2+3^2}}\right\rvert=\frac{14}{\sqrt{19}}$ Projection/Resolving : $\overrightarrow{BA}=\begin{pmatrix}1\\2\\3\end{pmatrix}\Rightarrow\frac{\begin{pmatrix}1\\2\\3\end{pmatrix}.\begin{pmatrix}-1\\3\\3\end{pmatrix}}{\sqrt{(-1)^2+3^2+3^2}}=\frac{14}{\sqrt{19}}$		
	$V=\frac{1}{3}\times\frac{65\sqrt{19}}{2\sqrt{262}}\times\frac{14}{\sqrt{19}}=\ldots=\frac{455}{3\sqrt{262}}$ or $\frac{455\sqrt{262}}{786}$	M1: Uses $\frac{1}{3}\times$area $\Delta\times$perp.height and obtains a positive value. $\frac{1}{2}$ must have been used for triangle area earlier unless they now use $\frac{1}{6}\times\ldots$ **Requires previous M mark.** A1: Either correct **exact** value	dM1 A1
			(5)
			Total 12

Question Number	Scheme	Notes	Marks
5	$\mathbf{M}=\begin{pmatrix}1 & 2 & k\\ -1 & -3 & 4\\ 2 & 6 & -8\end{pmatrix}$		
(i) & (ii) Mark the parts together	$\det\begin{pmatrix}1-\lambda & 2 & k\\ -1 & -3-\lambda & 4\\ 2 & 6 & -8-\lambda\end{pmatrix}$ $=\pm\left[(1-\lambda)((-3-\lambda)(-8-\lambda)-24)-2((-1)(-8-\lambda)-8)+k((-1)(6)-2(-3-\lambda))\right]$	Recognisable complete attempt at $\det(\mathbf{M}-\lambda\mathbf{I})$. May use other rows/columns. Allow $\pm$ and slips including +2 for first –2	M1
	Sarrus $\Rightarrow\pm[(1-\lambda)(-3-\lambda)(-8-\lambda)+(2)(4)(2)+(k)(-1)(6)-(k)(-3-\lambda)(2)-(1-\lambda)(4)(6)-(2)(-1)(-8-\lambda)]$		
	$=(1-\lambda)(\lambda^2+11\lambda)-2\lambda+2k\lambda$ $=-\lambda^3-10\lambda^2+9\lambda+2k\lambda$ $=\lambda(-\lambda^2-10\lambda+9+2k)$	M1: Obtains $\{\lambda\}(a\lambda^2+b\lambda+c+dk \text{ oe})$ $a,\ b,\ c,\ d\neq 0$ A1: Correct expression – allow: $\pm\{\lambda\}(-\lambda^2-10\lambda+9+2k \text{ oe})$ or $\pm\{\lambda\}(\lambda^2+10\lambda-9-2k \text{ oe})$ Allow quadratic to be unsimplified and the marks can be implied if the initial λ has been removed	M1 A1
	{One eigenvalue is zero, if repeated then} $9+2k=0\Rightarrow k=\ldots$ **or** $\left\{\pm(-\lambda^2-10\lambda+9+2k)\text{ has repeated roots so}\right\}$ $b^2-4ac=0\Rightarrow\begin{cases}100-4(-1)(9+2k)=0\\100-4(1)(-9-2k)=0\end{cases}\Rightarrow k=\ldots$	Attempts to set their $c+dk=0$ and solves for k **or** Considers the case of their quadratic $a\lambda^2+b\lambda+c+dk=0$ having a repeated root and uses a valid strategy to find k	M1
	Alternative approaches with $\lambda^2+10\lambda-9-2k=0$: $(\lambda+a)^2=\lambda^2+2a\lambda+a^2\Rightarrow 2a=10\Rightarrow -9-2k=5^2\Rightarrow k=\ldots$ sum of roots $=-10\Rightarrow$ root $=-5\Rightarrow$ product of roots $=(-5)^2=-9-2k\Rightarrow k=\ldots$		
	$k=-\dfrac{9}{2}$ or $k=-17$	One correct value for k	A1
	{One eigenvalue is zero, if repeated then} $9+2k=0\Rightarrow k=\ldots$ **and** $\left\{\pm(-\lambda^2-10\lambda+9+2k)\text{ has repeated roots so}\right\}$ $b^2-4ac=0\Rightarrow\begin{cases}100-4(-1)(9+2k)=0\\100-4(1)(-9-2k)=0\end{cases}\Rightarrow k=\ldots$	Attempts to set their $c+dk=0$ and solves for k **and** Considers the case of their quadratic $a\lambda^2+b\lambda+c+dk=0$ having a repeated root and uses a valid strategy to find k	M1
	$k=-\dfrac{9}{2}$ with eigenvalue -10 {and 0 repeated} $k=-17$ with eigenvalue -5 {repeated and 0}	Both correct values of k and the associated non-zero eigenvalues clearly assigned. No additional eigenvalues or values for k	A1
			Total 7

Question Number	Scheme	Notes	Marks
	$\frac{x^2}{16}+\frac{y^2}{9}=1 \qquad P(4\cos\theta,\ 3\sin\theta)$		
6(a)	$\frac{dy}{dx}=-\frac{3\cos\theta}{4\sin\theta}$ or $\frac{2x}{16}+\frac{2y}{9}\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{18x}{32y}$ or $\frac{x^2}{16}+\frac{y^2}{9}=1\Rightarrow y=3\left(1-\frac{x^2}{16}\right)^{\frac{1}{2}}\Rightarrow\frac{dy}{dx}=\frac{3}{2}\left(1-\frac{x^2}{16}\right)^{-\frac{1}{2}}\times-\frac{2x}{16}$	Uses a correct method and finds an expression for $\frac{dy}{dx}$ of the correct form (sign and coefficient slips only)	M1
	$\frac{dy}{dx}=-\frac{3\cos\theta}{4\sin\theta}$ oe e.g. $-\frac{3}{4}\cot\theta$ oe	Any correct derivative in terms of θ only.	A1
	$y-3\sin\theta=-\frac{3\cos\theta}{4\sin\theta}(x-4\cos\theta)$ or or $y=-\frac{3\cos\theta}{4\sin\theta}x+c\Rightarrow 3\sin\theta=-\frac{3\cos\theta}{4\sin\theta}4\cos\theta+c$ $\Rightarrow c=\ldots\left\{\frac{12\sin^2\theta+12\cos^2\theta}{4\sin\theta}\right\}$	Applies correct straight line method using any gradient in terms of θ. If they use $y=mx+c$ they must substitute coordinates correctly and reach $c=\ldots$ M0 if use normal gradient	M1
	$\Rightarrow 4y\sin\theta-12\sin^2\theta=-3x\cos\theta+12\cos^2\theta$ or using $y=mx+c: y=-\frac{3\cos\theta}{4\sin\theta}x+12\Rightarrow 4y\sin\theta=-3x\cos\theta+12$ $\Rightarrow 3x\cos\theta+4y\sin\theta\left\{=12(\cos^2\theta+\sin^2\theta)\right\}=12$ M1: Multiplies through to remove fraction to obtain an equation with trig expressions in sin and cos only. Allow this mark if they go straight to the given answer from a correct equation. Can score from use of a normal gradient and/or with coordinates wrongly placed but there must have been an attempt at a line. A1*: Correct equation from correct work. $\sin^2\theta$ and $\cos^2\theta$ must be seen somewhere in the working. Accept e.g., $\sin^2\theta+\cos^2\theta=1$ seen in side-working		M1 A1*
			(5)
(b)	$y-3\sin\theta=\frac{4\sin\theta}{3\cos\theta}(x-4\cos\theta)$ oe e.g., $4x\sin\theta-3y\cos\theta=7\sin\theta\cos\theta$ or $y=\frac{4\sin\theta}{3\cos\theta}x+c$ $\Rightarrow 3\sin\theta=\frac{4\sin\theta}{3\cos\theta}4\cos\theta+c\Rightarrow c=\ldots\left\{\frac{-7\sin\theta\cos\theta}{3\cos\theta}\right\}$	M1: Applies correct straight line method with the negative reciprocal of their tangent gradient. If $y=mx+c$ is used coordinates must be substituted correctly and $c=\ldots$ reached A1: Any correct equation	M1 A1
			(2)

Question Number	Scheme	Notes	Marks
6(c)	A is $\left(\dfrac{4}{\cos\theta},0\right)$	Any correct x-axis intercept of the tangent. Allow e.g., $\{x=\}\dfrac{12}{3\cos\theta},\ 4\sec\theta$ Could be on a diagram or implied by midpoint	B1
	$x=0\Rightarrow y-3\sin\theta=-\dfrac{16}{3}\sin\theta\Rightarrow B$ is $\left(0,-\dfrac{7}{3}\sin\theta\right)$	Sets $x = 0$ in their **normal** equation (changed gradient) and finds y. Could be implied. Allow just $-\dfrac{7}{3}\sin\theta$ oe	M1
	So midpoint M of AB is $\left(\dfrac{2}{\cos\theta},-\dfrac{7}{6}\sin\theta\right)$	Any correct midpoint. Accept any equivalents and as $x=\ldots,\ y=\ldots$	A1
	$\sin^2\theta+\cos^2\theta=1\Rightarrow\left(-\dfrac{6}{7}y\right)^2+\left(\dfrac{2}{x}\right)^2=1$	Uses $\sin^2\theta+\cos^2\theta=1$ to obtain an equation in x and y only. May follow incorrect or no attempt at midpoint	M1
	$\Rightarrow\dfrac{36}{49}y^2+\dfrac{4}{x^2}=1\Rightarrow 36x^2y^2+49\times4=49x^2$ $\Rightarrow x^2\left(49-36y^2\right)=196$	**d**M1: Rearranges to the form $x^2\left(p\pm qy^2\right)=r,\quad p,\ q,\ r\in\mathbb{Z}$ **Requires all previous M marks.** A1: Correct equation	**d**M1 A1
			(6)
	Note that is possible to use e.g., $1+\tan^2\theta=\sec^2\theta$, for example: $M\left(2\sec\theta,\dfrac{-7\tan\theta}{6\sec\theta}\right)\Rightarrow\sec\theta=\dfrac{x}{2},\ y=\dfrac{-7\tan\theta}{3x}\Rightarrow\tan\theta=\dfrac{-3xy}{7}\Rightarrow1+\dfrac{9x^2y^2}{49}=\dfrac{x^2}{4}$ (2nd M1) $\Rightarrow1+\dfrac{9x^2y^2}{49}=\dfrac{x^2}{4}\Rightarrow196+36x^2y^2=49x^2\Rightarrow x^2\left(49-36y^2\right)=196$ (3rd M1, A1)		**Total 13**

Question Number	Scheme	Notes	Marks
7(a) **Way 1**	$I_n = \int \cosh^n 2x\ \mathrm{d}x = \int \cosh 2x \cosh^{n-1} 2x\ \mathrm{d}x$ $= \frac{1}{2}\sinh 2x \cosh^{n-1} 2x - \int \frac{1}{2}\sinh 2x \times (n-1)\cosh^{n-2} 2x \times 2\sinh 2x\,\mathrm{d}x$	M1: Correct split and attempts to apply parts to obtain an expression of the correct form (sign and coefficient errors only). A1: Any correct expression	M1 A1
	$\left\{= \frac{1}{2}\sinh 2x \cosh^{n-1} 2x - (n-1)\int \sinh^2 2x \cosh^{n-2} 2x\,\mathrm{d}x\right\}$ $= \frac{1}{2}\sinh 2x \cosh^{n-1} 2x - (n-1)\int \left(\cosh^2 2x - 1\right)\cosh^{n-2} 2x\,\mathrm{d}x$	Applies $\sinh^2 2x = \pm\cosh^2 2x \pm 1$ **Requires previous M mark.**	**d**M1
	$\Rightarrow I_n = \frac{1}{2}\sinh 2x \cosh^{n-1} 2x - (n-1)\left(I_n - I_{n-2}\right)$	Introduces I_n and I_{n-2} - not implied by given answer. **Requires previous M mark.**	**dd**M1
	$\left\{\Rightarrow nI_n = \frac{1}{2}\sinh 2x \cosh^{n-1} 2x + (n-1)I_{n-2}\right\}$ $I_n = \frac{\sinh 2x \cosh^{n-1} 2x}{2n} + \frac{n-1}{n}I_{n-2}$ *	Fully correct proof. Condone missing '$\mathrm{d}x$'s. Poor bracketing must be recovered before given answer but no other errors e.g., sin for sinh, or wrong or missing arguments	A1*
	Accept e.g., $I_n = \frac{(n-1)I_{n-2}}{n} + \frac{1}{2n}\sinh 2x \cosh^{n-1} 2x$		**(5)**
Way 2	$I_n = \int \cosh^n 2x\ \mathrm{d}x = \int \cosh^2 2x \cosh^{n-2} 2x\ \mathrm{d}x$ $= \int \left(\sinh^2 2x + 1\right)\cosh^{n-2} 2x\ \mathrm{d}x$	M1: Correct split and applies $\sinh^2 2x = \pm\cosh^2 2x \pm 1$ to obtain an expression of the correct form (sign and coefficient errors only). A1: Correct expression	M1 A1
	$\left\{= \int \cosh^{n-2} 2x\ \mathrm{d}x + \int \sinh^2 2x \cosh^{n-2} 2x\ \mathrm{d}x\right\}$ $\int \sinh^2 2x \cosh^{n-2} 2x\ \mathrm{d}x \left\{= \int \sinh 2x \cosh^{n-2} 2x \sinh 2x\ \mathrm{d}x\right\}$ $= \frac{1}{2\left(n-1\right)}\sinh 2x \cosh^{n-1} 2x - \frac{1}{n-1}\int \cosh^n 2x\ \mathrm{d}x$	Attempts to apply parts to obtain an expression of the correct form for $\int \sinh^2 2x \cosh^{n-2} 2x\ \mathrm{d}x$ **Requires previous M mark.**	**d**M1
	$\Rightarrow I_n = I_{n-2} + \frac{1}{2\left(n-1\right)}\sinh 2x \cosh^{n-1} 2x - \frac{1}{n-1}I_n$	Introduces I_n and I_{n-2} - not implied by given answer. **Requires previous M mark.**	**dd**M1
	$\left\{\Rightarrow \left(n-1\right)I_n = \frac{1}{2}\sinh 2x \cosh^{n-1} 2x + (n-1)I_{n-2} - I_n\right\}$ $I_n = \frac{\sinh 2x \cosh^{n-1} 2x}{2n} + \frac{n-1}{n}I_{n-2}$ *	Fully correct proof. Condone missing '$\mathrm{d}x$'s. Poor bracketing must be recovered before given answer but no other errors e.g., sin for sinh, or wrong or missing arguments	A1*
	Accept e.g., $I_n = \frac{(n-1)I_{n-2}}{n} + \frac{1}{2n}\sinh 2x \cosh^{n-1} 2x$		**(5)**

Question Number	Scheme	Notes	Marks
7(b)	$(1+\cosh 2x)^3 = 1+3\cosh 2x+3\cosh^2 2x+\cosh^3 2x$ Correct expansion. Could be implied e.g. by $x+3I_1+3I_2+I_3$ and allow if correct but terms are not collected. Condone if partially or completely in "x" provided terms <u>are</u> collected		B1
	$\int \cosh^2 2x\,\mathrm{d}x$ or $I_2=\frac{1}{4}\sinh 2x\cosh 2x+\frac{1}{2}I_0$ or $\int \cosh^3 2x\,\mathrm{d}x$ or $I_3=\frac{1}{6}\sinh 2x\cosh^2 2x+\frac{2}{3}I_1$	Completes an attempt to apply the reduction formula for I_2 **or** I_3. May be slips but must get two terms. May be seen with I_0 / I_1 attempted and/or embedded in expression for $\int(1+\cosh 2x)^3\,\mathrm{d}x$	M1
	$I_0=x\qquad I_1=\frac{1}{2}\sinh 2x$ $\int(1+\cosh 2x)^3\,\mathrm{d}x=\int(1+3\cosh 2x)\mathrm{d}x+3I_2+I_3=$ $x+\frac{3}{2}\sinh 2x+\frac{3}{4}\sinh 2x\cosh 2x+\frac{3}{2}x+\frac{1}{6}\sinh 2x\cosh^2 2x+\frac{1}{3}\sinh 2x(+c)$	$I_0=x$ **and** $I_1=\pm k\sinh 2x$ (condone I_1 from formula) **and** $\int(1+3\cosh 2x)\mathrm{d}x\to x\pm q\sinh 2x$ **and** uses the above to obtain an expression for $\int(1+\cosh 2x)^3\,\mathrm{d}x$ **Requires previous M mark.**	dM1
	Note: **One** of I_2 and I_3 may be attempted directly – if so correct identities must be used and an expression of a correct form obtained. Examples: $I_2=\int\cosh^2 2x\,\mathrm{d}x=\int\left(\frac{1}{2}\cosh 4x+\frac{1}{2}\right)\mathrm{d}x=\frac{1}{8}\sinh 4x+\frac{x}{2}$ $\Rightarrow x+\frac{3}{2}\sinh 2x+\frac{3}{8}\sinh 4x+\frac{3}{2}x+\frac{1}{6}\sinh 2x\cosh^2 2x+\frac{1}{3}\sinh 2x(+c)$ $I_3=\int\cosh^3 2x\,\mathrm{d}x=\int\cosh 2x(\sinh^2 2x+1)\mathrm{d}x=\frac{1}{6}\sinh^3 2x+\frac{1}{2}\sinh 2x$ $\Rightarrow x+\frac{3}{2}\sinh 2x+\frac{3}{4}\sinh 2x\cosh 2x+\frac{3}{2}x+\frac{1}{6}\sinh^3 2x+\frac{1}{2}\sinh 2x(+c)$ If exponential definitions are used they must be correct.		
	$=\frac{5}{2}x+\frac{11}{6}\sinh 2x+\frac{3}{4}\sinh 2x\cosh 2x+\frac{1}{6}\sinh 2x\cosh^2 2x(+c)$	Correct answer. Award when a correct expression with collected like terms is seen.	A1
	I_2 attempted directly $\Rightarrow\frac{5}{2}x+\frac{11}{6}\sinh 2x+\frac{3}{8}\sinh 4x+\frac{1}{6}\sinh 2x\cosh^2 2x(+c)$ I_3 attempted directly $\Rightarrow\frac{5}{2}x+2\sinh 2x+\frac{3}{4}\sinh 2x\cosh 2x+\frac{1}{6}\sinh^3 2x(+c)$		**(4)**
	If identities are used before a correct answer is seen with like terms collected then the work must be correct		**Total 9**

Question Number	Scheme	Notes	Marks
8(a)	$\left\{\frac{dy}{dx}=\right\}\text{arcosh}\,5x+\frac{ax}{\sqrt{bx^2-1}}$ or $\text{arcosh}\,5x+\frac{cx}{\sqrt{x^2-d}}$ (M1) $\Rightarrow \text{arcosh}(5x)+\frac{5x}{\sqrt{25x^2-1}}$ (A1) M1: Differentiates to obtain expression of the correct form $a,\ b,\ c,\ d\neq 0$ A1: Correct differentiation. Any equivalent form.		M1 A1
			(2)
(b)	$\frac{d}{dx}\left(x\text{arcosh}(5x)\right)=\text{arcosh}(5x)+"\frac{5x}{\sqrt{25x^2-1}}"\Rightarrow\int\text{arcosh}(5x)\,dx=x\,\text{arcosh}(5x)-\int"\frac{5x}{\sqrt{25x^2-1}}"dx$ M1: Rearranges **their** answer to (a) correctly and integrates or uses the correct formula to apply parts to $1\times\text{arcosh}\,5x$ to obtain the above.		M1
	$\int\text{arcosh}(5x)\,dx=x\,\text{arcosh}(5x)-\int\frac{5x}{\sqrt{25x^2-1}}dx$ A1: **Correct** expression – but **see note below on limited ft**		A1 **(limited ft)**
	$=x\,\text{arcosh}(5x)-\frac{1}{5}\left(25x^2-1\right)^{\frac{1}{2}}\ (+c)$	M1: $\int\frac{Ax}{\sqrt{Bx^2-1}}dx\to C\left(Bx^2-1\right)^{\frac{1}{2}}$ A1: Fully correct expression with $x$arcosh($5x$) - see note below for limited ft	M1 A1 **(limited ft)**
	Note: Substitutions: $u=5x\Rightarrow\left(u^2-1\right)^{\frac{1}{2}}\Rightarrow\left[\frac{1}{5}\sqrt{u^2-1}\right]_{\frac{5}{4}}^{3}$ $\quad u=25x^2-1\Rightarrow\left[\frac{1}{5}\sqrt{u}\right]_{\frac{9}{16}}^{8}$ M1: Correct form A1: Fully correct expression with $x$arcosh($5x$)		
	A limited ft for **one** of the errors in (a) shown below applies for the first two A marks. **However also allow the following if this error occurs in part (b)** which is most likely to come from not rearranging and effectively restarting by using parts. Note that substitutions could be used. $a=1\Rightarrow x\,\text{arcosh}(5x)-\int\frac{x}{\sqrt{25x^2-1}}dx\Rightarrow x\,\text{arcosh}(5x)-\frac{1}{25}\left(25x^2-1\right)^{\frac{1}{2}}\ (+c)$ $b=5\Rightarrow x\,\text{arcosh}(5x)-\int\frac{5x}{\sqrt{5x^2-1}}dx\Rightarrow x\,\text{arcosh}(5x)-\left(5x^2-1\right)^{\frac{1}{2}}\ (+c)$ $a=-5\Rightarrow x\,\text{arcosh}(5x)+\int\frac{5x}{\sqrt{25x^2-1}}dx\Rightarrow x\,\text{arcosh}(5x)+\frac{1}{5}\left(25x^2-1\right)^{\frac{1}{2}}\ (+c)$		
	$\int_{\frac{1}{4}}^{\frac{3}{5}}\text{arcosh}\,5x\,dx=\frac{3}{5}\text{arcosh}(3)-\frac{1}{5}\sqrt{25\times\frac{9}{25}-1}-\left(\frac{1}{4}\text{arcosh}\left(\frac{5}{4}\right)-\frac{1}{5}\sqrt{25\times\frac{1}{16}-1}\right)$ Applies appropriate limits (note substitutions above) with subtraction the right way round seen to obtain an expression of the form $x\,\text{arcosh}(5x)\pm f(x)$ **where f(*x*) has come from integration**		M1
	$=\frac{3}{5}\text{arcosh}(3)-\frac{2\sqrt{2}}{5}-\frac{1}{4}\text{arcosh}\left(\frac{5}{4}\right)+\frac{3}{20}$	Correct answer seen in any form. **Must not follow clearly incorrect work.**	A1
	$\text{arcosh}3=\ln\left(3+\sqrt{3^2-1^2}\right)$ or $\text{arcosh}\left(\frac{5}{4}\right)=\ln\left(\frac{5}{4}+\sqrt{\left(\frac{5}{4}\right)^2-1^2}\right)$ $\left\{\Rightarrow\frac{3}{5}\ln\left(3+\sqrt{8}\right)-\frac{2\sqrt{2}}{5}-\frac{1}{4}\ln 2+\frac{3}{20}\right\}$	Converts $\text{arcosh}(3)$ or $\text{arcosh}\left(\frac{5}{4}\right)$ to any correct log form. Independent mark but must have obtained $x\,\text{arcosh}(5x)\pm f(x)$ **where f(*x*) has come from integration**	M1
	$=\frac{3}{20}-\frac{2\sqrt{2}}{5}+\ln\left(3+2\sqrt{2}\right)^{\frac{3}{5}}-\frac{1}{4}\ln 2$ **Must not follow clearly incorrect work.**	Correct answer. Terms in any order but otherwise written as shown. Allow values for p, q, r & k	A1
			(8)
			Total 10
	PAPER TOTAL: 75		

Question Number	Scheme	Notes	Marks
1(i)	$(8)\int \frac{1}{16+x^2}\,dx=(8)\left(\frac{1}{4}\arctan\left(\frac{x}{4}\right)\right)$	Obtains ...arctan (kx) Allow k = 1	M1
	$2\left[\arctan\left(\frac{x}{4}\right)\right]_4^{4\sqrt{3}}=2(\arctan\sqrt{3}-\arctan 1)=...$	Substitutes the given limits, subtracts either way round and obtains a value (could be a decimal). The substitution does not need to be seen explicitly and may be implied by their value.	**d**M1
	$\frac{\pi}{6}$ or $p=\frac{1}{6}$ Correct exact value (or value for p) Accept equivalent exact expressions e.g. $\frac{2\pi}{12}$ or $p=\frac{2}{12}$ and isw if necessary.		A1
			(3)
(ii)	$2\int \frac{1}{\sqrt{9-4x^2}}\,dx=2\left(\frac{1}{2}\arcsin\frac{2x}{3}\right)\left(\text{or e.g. }\arcsin\frac{x}{3/2}\right)$ **M1**: Obtains ...arcsin (kx). Allow k = 1 so allow just arcsin x. **A1**: Fully correct integration but allow unsimplified as above		M1 A1
	$\left[\arcsin\left(\frac{2x}{3}\right)\right]_{\frac{3}{4}}^{k}=\arcsin\left(\frac{2k}{3}\right)-\arcsin\left(\frac{1}{2}\right)=\frac{\pi}{12}$ $\Rightarrow \arcsin\left(\frac{2k}{3}\right)=\frac{\pi}{12}+\frac{\pi}{6}\Rightarrow\frac{2k}{3}=\sin\left(\frac{\pi}{4}\right)\Rightarrow\frac{2k}{3}=\frac{\sqrt{2}}{2}\Rightarrow k=...$ Substitutes the given limits, subtracts either way round, sets = $\frac{\pi}{12}$, uses $\arcsin\left(\frac{1}{2}\right)=\frac{\pi}{6}$ and the correct **order** of operations condoning sign errors only to reach a value for k e.g. $\pm\alpha\left(\arcsin\left(\frac{2k}{3}\right)-\frac{\pi}{6}\right)=\frac{\pi}{12}\Rightarrow\arcsin\left(\frac{2k}{3}\right)=\frac{\pi}{12\alpha}\pm\frac{\pi}{6}\Rightarrow k=\frac{3\sin\left(\frac{\pi}{12\alpha}\pm\frac{\pi}{6}\right)}{2}$ Note that k may be inexact (decimal) or may be in terms of "sin" but must have a simplified argument e.g. $k=\frac{3\sin\left(\frac{\pi}{4}\right)}{2}$		**d**M1
	$k=\frac{3\sqrt{2}}{4}$ or exact equivalent e.g., $\frac{3}{2\sqrt{2}}$ Note that a common incorrect answer is $k=\frac{3}{2}\sin\left(\frac{5\pi}{24}\right)(=0.913...)$ which comes from an incorrect integral of $2\arcsin\left(\frac{2x}{3}\right)$ (generally scoring 1010) Condone $x=\frac{3\sqrt{2}}{4}$		A1
			(4)
			Total 7

Question Number	Scheme	Notes	Marks
2(a) **Way 1** **TU = I**	**TU = I** $\Rightarrow \begin{pmatrix} 2 & 3 & 7 \\ 3 & 2 & 6 \\ a & 4 & b \end{pmatrix}\begin{pmatrix} 6 & -1 & -4 \\ 15 & c & -9 \\ -8 & a & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\Rightarrow$ e.g., $\begin{matrix} 6a+60-8b=0 \\ -4a-36+5b=1 \end{matrix}$ or $\begin{matrix} -2+3c+7a=0 \\ -3+2c+6a=1 \end{matrix}$ Obtains at least 2 equations with at least one correct. (condone column × row multiplication leading to the way 2 equations – see below). Ignore errors in unused elements or equations.		M1
	e.g., $\begin{matrix} 6a-8b=-60 \\ -4a+5b=37 \end{matrix} \Rightarrow a=\ldots,\ b=\ldots$ **or** $\begin{matrix} 7a+3c=2 \\ 6a+2c=4 \end{matrix} \Rightarrow a=\ldots,\ c=\ldots$ Obtains values for two of *a*, *b* and *c*. You do **not** need to check their values. As long as the previous M mark was scored, it is sufficient to just write down values.		**d**M1
	$a=2,\ b=9,\ c=-4$	A1: Two correct values A1: All three correct values and no extra values unless they are rejected.	A1 A1
			(4)
Way 2 **UT = I** **For first 2 marks**	**UT = I** $\Rightarrow \begin{pmatrix} 6 & -1 & -4 \\ 15 & c & -9 \\ -8 & a & 5 \end{pmatrix}\begin{pmatrix} 2 & 3 & 7 \\ 3 & 2 & 6 \\ a & 4 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $12-3-4a=1$ $\Rightarrow$ e.g., $42-6-4b=0$ $[45+2c-36=1]$ Obtains at least 2 equations with at least one correct. (condone column × row multiplication leading to the way 1 equations – see above). Ignore errors in unused elements or equations.		M1
	e.g., $-4a=-8,\ -4b=-36\ \ [2c=\ -8]$ $\Rightarrow a=\ldots,\ b=\ldots$ Obtains values for two of *a*, *b* and *c*. You do **not** need to check their values. As long as the previous M mark was scored, it is sufficient to just write down values.		**d**M1

Way 3 **Inverses** **For first mark**	$\mathbf{T}^{-1}=\mathbf{U}\Rightarrow\frac{1}{4a-5b+36}\begin{pmatrix}2b-24 & -3b+28 & 4\\ 6a-3b & -7a+2b & 9\\ -2a+12 & 3a-8 & -5\end{pmatrix}=\begin{pmatrix}6 & -1 & -4\\ 15 & c & -9\\ -8 & a & 5\end{pmatrix}$ $\Rightarrow \text{e.g.,}\ \frac{4}{4a-5b+36}=-4,\ \frac{2b-24}{4a-5b+36}=6\ \left[\frac{-7a+2b}{4a-5b+36}=c\right]$ For $\mathbf{T}^{-1}=\frac{1}{\mathrm{f}(a,b)}\mathbf{M}$ where **M** has at least 1 correct element **and** obtains 2 equations. Note that there is no requirement to find all the elements of **M**. **OR** $\mathbf{U}^{-1}=\mathbf{T}\Rightarrow\frac{1}{-6a-2c+3}\begin{pmatrix}9a+5c & -4a+5 & 4c+9\\ -3 & -2 & -6\\ 15a+8c & -6a+8 & 6c+15\end{pmatrix}=\begin{pmatrix}2 & 3 & 7\\ 3 & 2 & 6\\ a & 4 & b\end{pmatrix}$ $\Rightarrow \text{e.g.,}\ \frac{-3}{-6a-2c+3}=3,\ \frac{4c+9}{-6a-2c+3}=7\ \left[\frac{6c+15}{-6a-2c+3}=b\right]$ For $\mathbf{U}^{-1}=\frac{1}{\mathrm{f}(a,c)}\mathbf{M}$ where **M** has at least 1 correct element **and** obtains 2 equations Note that there is no requirement to find all the elements of **M**.	M1

Question	Scheme		Marks
2(b)	$\frac{x-1}{3}=\frac{y}{-4}=z+2 \Rightarrow [l_2 : \mathbf{r} =] \begin{pmatrix}1\\0\\-2\end{pmatrix} \pm \lambda \begin{pmatrix}3\\-4\\1\end{pmatrix}$ or $\left(\mathbf{r}-\begin{pmatrix}1\\0\\-2\end{pmatrix}\right)\times\begin{pmatrix}3\\-4\\1\end{pmatrix}=\mathbf{0}$ Obtains parametric/vector form (allow one slip only) or clearly identifies position and direction vectors. May be implied by an attempt to transform both.		M1
	$\begin{pmatrix}6 & -1 & -4\\15 & '-4' & -9\\-8 & '2' & 5\end{pmatrix}\begin{pmatrix}1+3\lambda\\-4\lambda\\-2+\lambda\end{pmatrix}=\begin{pmatrix}6+18\lambda+4\lambda+8-4\lambda\\15+45\lambda+16\lambda+18-9\lambda\\-8-24\lambda-8\lambda-10+5\lambda\end{pmatrix}$ **or** their **U** × their $\begin{pmatrix}1 & 3\\0 & -4\\-2 & 1\end{pmatrix}$ **or** ×their $\begin{pmatrix}1\\0\\-2\end{pmatrix}$ **and** ×their $\begin{pmatrix}3\\-4\\1\end{pmatrix}$ **or** their **U** ×their $\begin{pmatrix}1\\0\\-2\end{pmatrix}$ **and U** ×e.g. $\begin{pmatrix}4\\-4\\-1\end{pmatrix}$ then $dir=\begin{pmatrix}32\\85\\-45\end{pmatrix}-\begin{pmatrix}14\\33\\-18\end{pmatrix}$ Complete and correct method with their *b* and *c* for their **U** × their parametric form or **U** × both vectors or **U** × 2 points on the line and attempts direction. Must be an attempt to mutliply correctly i.e. clearly not row×row but allow attempts that use $\mathbf{T}^{-1}$ for **U** using their *a* and *b* provided all elements are constants and it is a "changed" **T** **OR** $\begin{pmatrix}2 & 3 & 7\\3 & 2 & 6\\"2" & 4 & "9"\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+3\lambda\\-4\lambda\\-2+\lambda\end{pmatrix}\Rightarrow \begin{matrix}2x+3y+7z=1+3\lambda\\3x+2y+6z=-4\lambda\\2x+4y+9z=-2+\lambda\end{matrix}$ $x=18\lambda+14$ $\Rightarrow\ y=52\lambda+33$ $z=-18-27\lambda$ A complete method using their parametric form and their **T** to produce and solve 3 simultaneous equations to find *x*, *y* and *z* in terms of λ Alternatively solves $\mathbf{T}\mathbf{x}=("\mathbf{i}-2\mathbf{k}")$ and $\mathbf{T}\mathbf{x}=("3\mathbf{i}-4\mathbf{k}+\mathbf{k}")$ to find position and direction		M1
	$[l_1 : \mathbf{r} =]\begin{pmatrix}14+18\lambda\\33+52\lambda\\-18-27\lambda\end{pmatrix}$ $\Rightarrow \frac{x-14}{18}=\frac{y-33}{52}=\frac{z+18}{-27}$	**d**M1: Correctly converts their result into Cartesian equation. **Requires previous method mark** A1: Correct Cartesian equation - allow equivalents e.g., $\ldots=\frac{z-(-18)}{-27},\quad \ldots=\frac{-z-18}{27}$	**d**M1 A1
			(4)
			Total 8

2(b) Alternative

$$x = t \Rightarrow y = \frac{4}{3} - \frac{4}{3}t,\ \ z = \frac{1}{3}t - \frac{7}{3}$$

M1: Obtains parametric form (allow one slip only)

$$\begin{pmatrix} 6 & -1 & -4 \\ 15 & '-4' & -9 \\ -8 & '2' & 5 \end{pmatrix}\begin{pmatrix} t \\ \frac{4}{3} - \frac{4}{3}t \\ \frac{1}{3}t - \frac{7}{3} \end{pmatrix} = \begin{pmatrix} 6t - \frac{4}{3} + \frac{4}{3}t - \frac{4}{3}t + \frac{28}{3} \\ 15t - \frac{16}{3} + \frac{16}{3}t - 3t + 21 \\ -8t + \frac{8}{3} - \frac{8}{3}t + \frac{5}{3}t - \frac{35}{5} \end{pmatrix}$$

M1: As above

$$[l_1 : \mathbf{r} =]\begin{pmatrix} 8 + 6t \\ \frac{47}{3} + \frac{52}{3}t \\ -9 - 9t \end{pmatrix}$$

$$\Rightarrow \frac{x-8}{6} = \frac{y - \frac{47}{3}}{\frac{52}{3}} = \frac{z+9}{-9}$$

dM1A1: As above

Question Number	Scheme	Notes	Marks
3(a)(i)	$(\pm 7e, 0)$	Correct **coordinates** or $x = \pm 7e, y = 0$	B1
(ii)	$x = \pm\dfrac{7}{e}$	Correct **equations**	B1
	SC: If "49" used for "7" consistently in (i) and (ii) score B0 B1		
			(2)
(b)(i)	$(PS^2 =)(x - '7e')^2 + y^2$ oe e.g. $(PS^2 =)('7e' - x)^2 + y^2$	Correct expression or equivalent with their $7e$. Must be in terms of e, x and y only. Apply isw once a correct expression is seen.	B1ft
(ii)	$(PM^2 =)\left('\dfrac{7}{e}' - x\right)^2$ oe e.g. $\left(x - '\dfrac{7}{e}'\right)^2$	Correct expression or equivalent with their $\dfrac{7}{e}$. Must be in terms of e and x only. Apply isw once a correct expression is seen.	B1ft
			(2)
(c)	$\left(\dfrac{PS}{PM} = e \Rightarrow\right) PS^2 = e^2PM^2 \Rightarrow (x - '7e')^2 + y^2 = e^2\left('\dfrac{7}{e}' - x\right)^2$ $\Rightarrow x^2 - 14ex + 49e^2 + y^2 = 49 - 14ex + e^2x^2$ Applies $PS^2 = e^2PM^2$ with their PS and PM and expands (condone poor squaring)		M1
	$x^2(1-e^2) + y^2 = 49(1-e^2)$ $\Rightarrow \dfrac{x^2}{49} + \dfrac{y^2}{49(1-e^2)} = 1 \Rightarrow b^2 = 49(1-e^2)*$	Reaches given answer with fully correct proof. All shown steps required. Note that it is possible to obtain this result even if the B marks are not scored in (b) e.g. correct expressions but not in the forms required.	A1*
			(2)
(d)	$(4\sqrt{3})^2 = 49(1-e^2) \Rightarrow e^2 \ldots$ or $e = \ldots$	Replaces b^2 with $(4\sqrt{3})^2$ and solves for e^2 or e.	M1
	$e = \dfrac{1}{7}$	Correct exact value for e (Not $\pm$)	A1
			(2)

(e)	$x=\frac{7}{2}\Rightarrow\frac{\left(\frac{7}{2}\right)^2}{49}+\frac{y^2}{'48'}=1\Rightarrow y=...[(\pm)6]$	Substitutes into their ellipse equation and obtains a value for *y*	M1
	Area $\Delta OPM=\left(\frac{1}{2}\right)\left(\frac{7}{'\left(\frac{1}{7}\right)'}-\frac{7}{2}\right)('6')=...$ Correct method for area of triangle *OPM* with their $\frac{7}{e}$ and their 6 May see other approaches, e.g., "shoelace" method e.g. $\frac{1}{2}\begin{vmatrix}3.5 & 0 & 49 & 3.5\\ 6 & 0 & 6 & 6\end{vmatrix}=\frac{1}{2}(49\times6-6\times3.5)=...$		**d**M1
	$\frac{273}{2}$ or $136\frac{1}{2}$ or 136.5	Any correct exact value	A1
	Special Case: $x=\frac{7}{2}\Rightarrow\frac{\left(\frac{7}{2}\right)^2}{49}+\frac{y^2}{'48'}=1\Rightarrow y=36\Rightarrow$ Area $\Delta OPM=\left(\frac{1}{2}\right)\left(\frac{7}{'\left(\frac{1}{7}\right)'}-\frac{7}{2}\right)(36)=...(819)$ **Scores M0M1A0**		
			(3)
			Total 11

Question Number	Scheme	Notes	Marks
4(a)	$\mathbf{M}\boldsymbol{x}=\lambda\boldsymbol{x}\Rightarrow\begin{pmatrix}0&-1&3\\-1&4&-1\\3&-1&0\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}\lambda\\-2\lambda\\\lambda\end{pmatrix}\Rightarrow$ e.g., $2+3=\lambda\Rightarrow\lambda=5$ **or** $(\mathbf{M}-\lambda\mathbf{I})\boldsymbol{x}=0\Rightarrow\begin{pmatrix}-\lambda&-1&3\\-1&4-\lambda&-1\\3&-1&-\lambda\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow$ e.g., $-\lambda+2+3=0\Rightarrow\lambda=5$ M1: Correct method leading to a value for λ A1: Correct value Note that the working may be minimal so e.g. $2+3=\lambda\Rightarrow\lambda=5$ is sufficient. **Correct answer only scores both marks.**		M1 A1
			(2)
(b)	$\begin{pmatrix}0&-1&3\\-1&4&-1\\3&-1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-3\begin{pmatrix}x\\y\\z\end{pmatrix}$ **or** $\begin{pmatrix}3&-1&3\\-1&7&-1\\3&-1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ **or** e.g., $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\-1\\3\end{pmatrix}\times\begin{pmatrix}-1\\7\\-1\end{pmatrix}$ $\Rightarrow x=\ldots, y=\ldots, z=\ldots$ Uses $\mathbf{M}\boldsymbol{x}=-3\boldsymbol{x}$ or $(\mathbf{M}-(-3)\mathbf{I})\boldsymbol{x}=\mathbf{0}$ to produce simultaneous equations **and** obtains values for x, y and z (not all 0) **or** uses a suitable vector product (with two correct components if method unclear)		M1
	$k\begin{pmatrix}1\\0\\-1\end{pmatrix}$	Any correct eigenvector (allow $x=\ldots$, $y=\ldots$, $z=\ldots$ and apply isw if a vector is subsequently formed incorrectly)	A1
			(2)
(c)	$\mathbf{M}\boldsymbol{x}=\lambda\boldsymbol{x}\Rightarrow$ e.g., $-1(1)+3(1)=\lambda$ $(\mathbf{M}-\lambda\mathbf{I})x=0\Rightarrow$ e.g., $-\lambda-1+3=0$ **or** $\lambda^3-4\lambda^2-11\lambda+30=0$ $\det\mathbf{M}=-30=\lambda_1\lambda_2\lambda_3=-15\lambda$ $\lambda=2$ Correct value. May be seen in their **D** which may come from an attempt at $\mathbf{P}^{\mathrm{T}}\mathbf{MP}$.		B1
	$(\mathbf{D}=)\begin{pmatrix}-3&0&0\\0&'2'&0\\0&0&'5'\end{pmatrix}$	Diagonal matrix with –3 and their eigenvalues anywhere on the leading diagonal and 0's elsewhere. Ignore labelling.	B1ft
	$\begin{pmatrix}1\\-2\\1\end{pmatrix}\rightarrow\begin{pmatrix}\frac{\sqrt{6}}{6}\\-\frac{\sqrt{6}}{3}\\\frac{\sqrt{6}}{6}\end{pmatrix}$ or $'\begin{pmatrix}1\\0\\-1\end{pmatrix}'\rightarrow\begin{pmatrix}\frac{\sqrt{2}}{2}\\0\\-\frac{\sqrt{2}}{2}\end{pmatrix}$ or $\begin{pmatrix}1\\1\\1\end{pmatrix}\rightarrow\begin{pmatrix}\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\end{pmatrix}$ Correct method seen to normalise **at least one** eigenvector of the two given eigenvectors or their eigenvector from part (b). May be seen in their **P**.		M1
	$\mathbf{D}=\begin{pmatrix}-3&0&0\\0&2&0\\0&0&5\end{pmatrix}$ **and** $\mathbf{P}=\begin{pmatrix}\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}&\frac{\sqrt{6}}{6}\\0&\frac{\sqrt{3}}{3}&-\frac{\sqrt{6}}{3}\\-\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}&\frac{\sqrt{6}}{6}\end{pmatrix}$ Both fully correct, consistent and labelled matrices. Elements may not have had denominators rationalised. (Any columns of **P** could be in opposite direction)		A1
			(4)
			Total 8

Note that some candidates go straight into solving $|\mathbf{M}-\lambda\mathbf{I}|=0$ e.g.

$$\begin{vmatrix} -\lambda & -1 & 3 \\ -1 & 4-\lambda & -1 \\ 3 & -1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda\left(\lambda(\lambda-4)-1\right)+3+\lambda+3\left(1-3(4-\lambda)\right)=0$$

$$\Rightarrow \lambda^3-4\lambda^2-11\lambda+30=0 \Rightarrow \lambda=-3,\ 5,\ 2$$

If this is all they do then the B mark in (c) can be awarded for $\lambda = 2$

The other marks in the question are available for the appropriate work.

Question Number	Scheme	Notes	Marks
5(a) **Way 1** **From LHS**	$\left(1-\text{sech}^2 x=\right)1-\left(\dfrac{2}{e^x+e^{-x}}\right)^2$	Replaces sech x with correct expression in terms of exponentials	B1
	$=\dfrac{\left(e^x+e^{-x}\right)^2-4}{\left(e^x+e^{-x}\right)^2}=\dfrac{e^{2x}+2+e^{-2x}-4}{\left(e^x+e^{-x}\right)^2}$	Expresses as a single fraction (or 2 fractions with the same denominator) and expands numerator	M1
	$=\dfrac{\left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}=\tanh^2 x$	Fully correct proof	A1*
Way 2 **Diff. of 2 squares**	$1-\text{sech}^2 x=(1+\text{sech}\,x)(1-\text{sech}\,x)=\left(1+\dfrac{2}{e^x+e^{-x}}\right)\left(1-\dfrac{2}{e^x+e^{-x}}\right)$ Uses difference of two squares and replaces sech x with correct expression in terms of exponentials		B1
	$=\left(\dfrac{e^x+e^{-x}+2}{e^x+e^{-x}}\right)\left(\dfrac{e^x+e^{-x}-2}{e^x+e^{-x}}\right)=\dfrac{e^{2x}+1-2e^x+1+e^{-2x}-2e^{-x}+2e^x+2e^{-x}-4}{\left(e^x+e^{-x}\right)^2}$ Expresses as a single fraction and expands numerator		M1
	$=\dfrac{e^{2x}-2+e^{-2x}}{\left(e^x+e^{-x}\right)^2}=\dfrac{\left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}=\tanh^2 x$	Fully correct proof	A1*
Way 3 **From RHS**	$\left(\tanh^2 x=\right)\dfrac{\left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}$	Replaces tanh x with correct expression in terms of exponentials	B1
	$=\dfrac{e^{2x}-2+e^{-2x}}{\left(e^x+e^{-x}\right)^2}=\dfrac{e^{2x}+2+e^{-2x}}{\left(e^x+e^{-x}\right)^2}-\dfrac{4}{\left(e^x+e^{-x}\right)^2}$ Expands numerator and splits into two fractions		M1
	$=\dfrac{\left(e^x+e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}-\left(\dfrac{2}{e^x+e^{-x}}\right)^2=1-\text{sech}^2 x$	Fully correct proof	A1*
			(3)

<u>Allow "meet in the middle" approaches as long as a conclusion is given e.g. lhs = rhs</u>

Example:

$$rhs=\tanh^2 x=\frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}\quad\text{or}\quad lhs=1-\text{sech}^2 x=1-\left(\frac{2}{e^x+e^{-x}}\right)^2$$

B1: Replaces tanh x or sech x with a correct expression in terms of exponentials

$$\frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}=\frac{e^{4x}-2e^{2x}+1}{e^{4x}+2e^{2x}+1}\quad\text{and}\quad 1-\left(\frac{2}{e^x+e^{-x}}\right)^2=\frac{\left(e^x+e^{-x}\right)^2-4}{\left(e^x+e^{-x}\right)^2}=\frac{e^{2x}-2+e^{-2x}}{e^{2x}+2+e^{-2x}}$$

M1: Makes progress by e.g. removing brackets on *rhs* and expressing *lhs* as a single fraction and expands numerator

$$\frac{e^{2x}-2+e^{-2x}}{e^{2x}+2+e^{-2x}}=\frac{e^{4x}-2e^{2x}+1}{e^{4x}+2e^{2x}+1}\Rightarrow 1-\text{sech}^2 x=\tanh^2 x$$

A1: Correct proof and (minimal) conclusion e.g. "= rhs" etc.

$$1-\operatorname{sech}^2 x = 1-\left(\frac{2}{e^x+e^{-x}}\right)^2 = \frac{e^{2x}+2+e^{-2x}-4}{\left(e^x+e^{-x}\right)^2} = \frac{e^{2x}+e^{-2x}-2}{e^{2x}+e^{-2x}+2} = \frac{\sinh^2 x}{\cosh^2 x} = \tanh^2 x$$

$$1-\operatorname{sech}^2 x = 1-\left(\frac{2}{e^x+e^{-x}}\right)^2 = \frac{e^{2x}+2+e^{-2x}-4}{\left(e^x+e^{-x}\right)^2} = \frac{e^{2x}+e^{-2x}-2}{e^{2x}+e^{-2x}+2} = \tanh^2 x$$

Both score B1M1A0 as we would need to see numerators and denominators factorised.

Note that we will allow an equivalent identity to be proved by exponentials and the given identity deduced e.g.

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$$

B1: Correct exponential form seen for cosh or sinh used

$$= \frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4} - \frac{e^{2x}}{4} + \frac{1}{2} - \frac{e^{-2x}}{4} = 1$$

M1: Expands and collects terms

$$\Rightarrow \cosh^2 x - \sinh^2 x = 1 \Rightarrow 1-\operatorname{sech}^2 x = \tanh^2 x$$

A1*: Fully correct work leading to the correct identity

(b)	$\int \tanh^n 3x\,dx = \int \tanh^{n-2} 3x \tanh^2 3x\,dx$ $= \int \tanh^{n-2} 3x\left(1-\text{sech}^2 3x\right)dx$	Splits $\tanh^n 3x$ into $\tanh^{n-2} 3x \tanh^2 3x$ and applies $\tanh^2 3x = 1-\text{sech}^2 3x$	M1
	Do **not** condone $\int \tanh^n 3x\,dx = \int \tanh^{n-2} 3x \tanh^2 3x\,dx$ $= \int \tanh^{n-2} 3x\left(1-\text{sech}^2 x\right)dx$ unless it is clear that 3x was intended and is therefore recovered in subsequent work.		
	$= \int \tanh^{n-2} 3x\,dx - \int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx$ $\int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx = \frac{1}{3(n-1)}\tanh^{n-1} 3x$ Expands and integrates $\tanh^{n-2} 3x\,\text{sech}^2 3x$ to obtain $\alpha \tanh^{n-1} 3x$ **Or it is possible to use parts for** $\int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx$ **:** $\int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx = \frac{1}{3}\tanh 3x \tanh^{n-2} 3x - \frac{1}{3}\int 3(n-2)\tanh 3x \tanh^{n-3} 3x\,\text{sech}^2 3x\,dx$ $= \frac{1}{3}\tanh^{n-1} 3x - (n-2)\int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx$ $\Rightarrow \int \tanh^{n-2} 3x\,\text{sech}^2 3x\,dx = \frac{1}{3(n-1)}\tanh^{n-1} 3x$ To score it must be a complete method leading to $\alpha \tanh^{n-1} 3x$ as above		**d**M1
	$I_n = I_{n-2} - \frac{1}{3(n-1)}\left[\tanh^{n-1} 3x\right]_0^{\frac{1}{3}\ln 2} = I_{n-2} - \frac{1}{3(n-1)}\left(\frac{e^{2\ln 2}-1}{e^{2\ln 2}+1}\right)^{n-1}$ Introduces I_{n-2} and applies $x = \frac{1}{3}\ln 2$ using a correct exponential definition of tanh or accept use of a calculator if work is correct e.g. $\tanh(\ln 2) = \frac{3}{5}$		**dd**M1
	$I_n = I_{n-2} - \frac{\left(\frac{3}{5}\right)^{n-1}}{3(n-1)}$ but condone $I_n = I_{n-2} - \frac{\frac{3}{5}^{n-1}}{3(n-1)}$ Fully correct proof. Allow recovery from slips e.g. tanh →tan→tanh or e.g. 3x becoming x and then reverting to 3x again If there are clear errors that are not recovered score A0.		A1
			(4)

(c)	$I_5 = I_3 - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)} = I_1 - \frac{\left(\frac{3}{5}\right)^{3-1}}{3(3-1)} - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$ Uses their reduction formula to obtain I_5 in terms of I_1 Note that there may have already been an attempt at I_1 Condone the use of the letter p for the $\frac{3}{5}$ and allow a "made up" p for this mark. This may be implied by e.g. $I_5 = I_3 - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$, $I_3 = I_1 - \frac{\left(\frac{3}{5}\right)^{3-1}}{3(3-1)}$		M1
	$\int \tanh 3x \, dx = \frac{1}{3}\ln(\cosh 3x)$	Integrates to obtain $q\ln(\cosh rx)$ oe e.g. $q\ln(\text{sech } rx)$ Condone q and/or $r = 1$	M1
	$I_5 = \frac{1}{3}\ln\left(\frac{e^{\ln 2} + e^{-\ln 2}}{2}\right) - \frac{\left(\frac{9}{25}\right)}{6} - \frac{\left(\frac{81}{625}\right)}{12}$ Applies $x = \frac{1}{3}\ln 2$ using correct exponential definition of cosh or uses a calculator if work is correct e.g. $\cosh(\ln 2) = \frac{5}{4}$ to obtain a numerical expression for I_5 Must not be in terms of p now and must be using a value of p obtained in part (b)		**dd**M1
	$\frac{1}{3}\ln\frac{5}{4} - \frac{177}{2500}$	Correct answer in correct form (allow a =..., b =..., c =...) Allow –0.0708 for c	A1
			(4)
			Total 11

Note that part (c) is "Hence" so they need to be using the given reduction formula, however, it is possible to find I_3 directly e.g. :

$$I_5 = I_3 - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$$

$$\int \tanh^3 3x \, dx = \int \left(\tanh 3x - \tanh 3x \, \text{sech}^2 3x\right) dx = \left[\frac{1}{3}\ln(\cosh 3x) + \frac{1}{6}\text{sech}^2 3x\right]$$

Score **M1** for using the reduction formula to obtain I_5 in terms of I_3 (allow the letter p for the $\frac{3}{5}$ and allow a "made up" p for this mark) **and** then integrating $\tanh^3 3x$ to the correct form e.g. $\alpha\ln(\cosh 3x) + \beta\,\text{sech}^2 3x$ (oe)

The second **M** mark would also score at this point as in the main scheme for integrating $\tanh 3x$ to obtain $q\ln(\cosh rx)$ oe e.g. $q\ln(\text{sech } rx)$

$$\left[\frac{1}{3}\ln(\cosh 3x) + \frac{1}{6}\text{sech}^2 3x\right]_0^{\frac{1}{3}\ln 2} - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)} = \frac{1}{3}\ln\frac{5}{4} + \frac{1}{6}\times\frac{16}{25} - \frac{1}{6} - \frac{27}{2500}$$

ddM1 for a complete method **using both limits** to obtain a numerical expression for I_5 using the correct exponential definitions or via a calculator.

A1: $\frac{1}{3}\ln\frac{5}{4} - \frac{177}{2500}$

Correct answer in correct form

(allow a =..., b =..., c =...) Allow –0.0708 for c

Question Number	Scheme	Notes	Marks
6(a)	$\pm\overrightarrow{AB}=\pm\left(\begin{pmatrix}-1\\1\\3\end{pmatrix}-\begin{pmatrix}3\\2\\2\end{pmatrix}\right)=\pm\begin{pmatrix}-4\\-1\\1\end{pmatrix}, \pm\overrightarrow{AC}=\pm\left(\begin{pmatrix}-2\\4\\2\end{pmatrix}-\begin{pmatrix}3\\2\\2\end{pmatrix}\right)=\pm\begin{pmatrix}-5\\2\\0\end{pmatrix}, \pm\overrightarrow{BC}=\pm\left(\begin{pmatrix}-2\\4\\2\end{pmatrix}-\begin{pmatrix}-1\\1\\3\end{pmatrix}\right)=\pm\begin{pmatrix}-1\\3\\-1\end{pmatrix}$ Correct method to obtain two relevant vectors using **subtraction**. You can ignore labelling e.g. if they find $\overrightarrow{BA}$ but call it $\overrightarrow{AB}$		M1
	e.g., $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-4\\-1\\1\end{pmatrix}\times\begin{pmatrix}-5\\2\\0\end{pmatrix}=\begin{pmatrix}-2\\-5\\-13\end{pmatrix}$ Correct method to find the vector product of two relevant vectors (if a correct method is not shown, two correct components for their vectors must be obtained)		**d**M1
	e.g., $\begin{pmatrix}3\\2\\2\end{pmatrix}.\begin{pmatrix}2\\5\\13\end{pmatrix}=6+10+26=42$ Attempts the scalar product between their normal vector and any of the position vectors of *A*, *B* or *C*.		**dd**M1
	$2x+5y+13z=42$ oe e.g. $-2x-5y-13z+42=0$	Any correct **Cartesian** equation.	A1
			(4)
(a) alt 1	$\pm\overrightarrow{AB}=\pm\left(\begin{pmatrix}-1\\1\\3\end{pmatrix}-\begin{pmatrix}3\\2\\2\end{pmatrix}\right)=\pm\begin{pmatrix}-4\\-1\\1\end{pmatrix}, \pm\overrightarrow{AC}=\pm\left(\begin{pmatrix}-2\\4\\2\end{pmatrix}-\begin{pmatrix}3\\2\\2\end{pmatrix}\right)=\pm\begin{pmatrix}-5\\2\\0\end{pmatrix}, \pm\overrightarrow{BC}=\pm\left(\begin{pmatrix}-2\\4\\2\end{pmatrix}-\begin{pmatrix}-1\\1\\3\end{pmatrix}\right)=\pm\begin{pmatrix}-1\\3\\-1\end{pmatrix}$ Correct method to obtain two relevant vectors using subtraction.		M1
	e.g., $\mathbf{r}=\begin{pmatrix}3\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}-4\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}-5\\2\\0\end{pmatrix}\Rightarrow\begin{matrix}x=3-4\lambda-5\mu\\y=2-\lambda+2\mu\\z=2+\lambda\end{matrix}\Rightarrow\text{e.g. }\lambda=z-2$ Attempts the parametric equation of the plane and uses components to eliminate at least one of their parameters.		**d**M1
	e.g., $\begin{matrix}x=3-4\lambda-5\mu\\y=2-\lambda+2\mu\\z=2+\lambda\end{matrix}\Rightarrow\text{e.g. }\lambda=z-2\Rightarrow\text{e.g. }\mu=\frac{1}{2}(y-4+z)$ Eliminates both of their parameters.		**dd**M1
	e.g. $x=3-4(z-2)-\frac{5}{2}(y-4+z)$	Any correct **Cartesian** equation.	A1
(a) alt 2	$ax+by+cz=1\rightarrow\begin{matrix}3a+2b+2c=1\\-a+b+3c=1\\-2a+4b+2c=1\end{matrix}\Rightarrow a=\frac{1}{21},\ b=\frac{5}{42},\ c=\frac{13}{42}$ $\Rightarrow\frac{1}{21}x+\frac{5}{42}y+\frac{13}{42}z=1$ M1: Substitutes the given points to give 3 equations in 3 unknowns dM1: Solves simultaneously to find values for "*a*", "*b*" and "*c*" ddM1: Substitutes back in to obtain a Cartesian equation A1: Any correct equation		

(b)	Line DE : ($\mathbf{r}$ =) $\begin{pmatrix}-1\\1\\-2\end{pmatrix} \pm \lambda \begin{pmatrix}2\\5\\13\end{pmatrix}$	Obtains parametric form for line DE with their normal (or recalculated normal) seen or implied. Allow one slip only.	M1
	$14(2\lambda-1)-(5\lambda+1)-17(13\lambda-2)=-66 \Rightarrow \lambda=...$ Substitutes their parametric form into the equation of Π_2 and solves for λ – can follow M0 provided their parametric form was an attempt at $\overrightarrow{OD} \pm \lambda(\text{their } \mathbf{n})$		M1
	$\lambda = \dfrac{85}{198}$	A correct exact value for λ depending on their method e.g. use of $\mathbf{n}=-2\mathbf{i}-5\mathbf{j}-13\mathbf{k}$ gives $\lambda=-\dfrac{85}{198}$	A1
	$DE=\sqrt{(2\times\frac{85}{198})^2+(5\times\frac{85}{198})^2+(13\times\frac{85}{198})^2}$ or e.g. $E=\left(-\frac{14}{99},\frac{623}{198},\frac{709}{198}\right)\Rightarrow DE=\sqrt{\left(-1+\frac{14}{99}\right)^2+\left(1-\frac{623}{198}\right)^2+\left(-2-\frac{709}{198}\right)^2}$ Correct method to find a numerical expression for distance DE Requires previous method mark Note $DE=-\frac{85}{198}\sqrt{(2)^2+(5)^2+(13)^2}=...$ is ok for this mark		**d**M1
	$DE=\dfrac{85\sqrt{22}}{66}$	Correct exact answer in the required form or $p=\dfrac{85}{66}$ or $1\dfrac{19}{66}$ Not $DE=-\dfrac{85\sqrt{22}}{66}$	A1
			(5)

Beware – Special Case!

An incorrect sign of λ may fortuitously give the correct length DE.

E.g. $\begin{pmatrix}-1\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\5\\13\end{pmatrix}$ leading incorrectly to $\lambda=-\dfrac{85}{198}$ would lead in both **d**M1 cases above to $DE=\dfrac{85\sqrt{22}}{66}$

E.g. $\begin{pmatrix}-1\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}-2\\-5\\-13\end{pmatrix}$ leading incorrectly to $\lambda=\dfrac{85}{198}$ would lead in both **d**M1 cases above to $DE=\dfrac{85\sqrt{22}}{66}$

In such cases score as M1M1A0M1A1ft i.e. we will only penalise it once.

Way 2 **Sim. eqns** **For first three marks**	$(\pm)\left(\frac{x+1}{2}=\frac{y-1}{5}=\frac{z+2}{13}\right)$ $\Rightarrow y=\frac{5}{2}x+\frac{7}{2},\ z=\frac{13}{2}x+\frac{9}{2}$	Obtains Cartesian form for line *DE* with their normal (or recalculated normal) allowing one slip only **and** attempts to find two variables in terms of the other variable	M1
	$14x-\left(\frac{5}{2}x+\frac{7}{2}\right)-17\left(\frac{13}{2}x+\frac{9}{2}\right)=-66$ $\Rightarrow\ x=-\frac{14}{99},\ y=\frac{623}{198},\ z=\frac{709}{198}$	M1: Substitutes into the plane equation and finds *x* =..., *y* =..., *z* =... A1: Correct exact values $\Longrightarrow$ **Way 1** for last two marks	M1 A1
(c)	e.g. $\overrightarrow{AF}.\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}1\\1\\q-2\end{pmatrix}.\begin{pmatrix}2\\5\\13\end{pmatrix}=2+5+13q-26$ or e.g. $\begin{vmatrix}-4&-1&1\\-5&2&0\\1&1&q-2\end{vmatrix}=-4(2(q-2))-5(q-2)-5-2$ or e.g. rule of Sarrus: $\begin{vmatrix}-4&-1&1&-4&-1\\-5&2&0&-5&2\\1&1&q-2&1&1\end{vmatrix}=-4(2(q-2))-5-5(q-2)-2$ Correct method for vector between *F* and *A*, *B* or *C* and finds scalar product with their normal or attempts the scalar triple product to obtain a linear expression in *q*. For the scalar triple product look for at least 2 correct "elements".		M1
	$\frac{1}{6}(13q-19)=\pm12\Rightarrow q=...$ Sets $\frac{1}{6}$ of their expression in *q* equal to one or both of ±12 (or equivalent work e.g. their expression in *q* equal to one or both of ±72) and proceeds to a value for *q*		**d**M1
	$q=7,-\frac{53}{13}$	Correct values. Allow exact equivalents for $-\frac{53}{13}$ e.g. $-4\frac{1}{13}$	A1
			(3)
			Total 12

Question Number	Scheme/Notes			Marks
7(a)	$y = \arccos(\operatorname{sech} x)$			
	e.g.:	$\cos y = \operatorname{sech} x \Rightarrow$		
	$\frac{dy}{dx} = -\frac{(-\operatorname{sech} x \tanh x)}{\sqrt{1-\operatorname{sech}^2 x}}$	$-\sin y \frac{dy}{dx} = -\operatorname{sech} x \tanh x$ or, e.g., $-\sin y = -\operatorname{sech} x \tanh x \frac{dx}{dy}$	$\cos y = (\cosh x)^{-1} \Rightarrow$ $-\sin y \frac{dy}{dx} = -(\cosh x)^{-2} \sinh x$	M1
	Differentiates to obtain an equation in $\frac{dy}{dx}$ or $\frac{dx}{dy}$ of the correct form e.g. condone coefficient sign errors only.			
	$\frac{dy}{dx} = \frac{\operatorname{sech} x \tanh x}{\tanh x}$	$\sqrt{1-\operatorname{sech}^2 x}\frac{dy}{dx} = \operatorname{sech} x \tanh x$ $\Rightarrow \tanh x \frac{dy}{dx} = \operatorname{sech} x \tanh x$	$\sqrt{1-\operatorname{sech}^2 x}\frac{dy}{dx} = \frac{\sinh x}{\cosh^2 x}$ $\Rightarrow \tanh x \frac{dy}{dx} = \frac{\sinh x}{\cosh^2 x}$	**d**M1
	Uses correct identities to obtain an equation in $\frac{dy}{dx}$ or $\frac{dx}{dy}$ in terms of x only with no roots but accept $\sqrt{\tanh^2 x}$ as "no roots"			
	$\Rightarrow \frac{dy}{dx} = \operatorname{sech} x$	$\Rightarrow \frac{dy}{dx} = \operatorname{sech} x$	$\frac{dy}{dx} = \frac{\cosh x}{\sinh x} \cdot \frac{\sinh x}{\cosh^2 x}$ $\Rightarrow \frac{dy}{dx} = \operatorname{sech} x$	A1*
	Fully correct proof. An equation in $\frac{dy}{dx}$ or $\frac{dx}{dy}$ and exactly two different hyperbolic functions with no roots must be seen before the given answer but accept $\sqrt{\tanh^2 x}$ as "no roots" Withhold this mark for any mathematical error e.g., **clear** use of $\frac{d}{dx}(\arccos x) = +\frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}(\operatorname{sech} x) = +\operatorname{sech} x \tanh x$ or e.g. hyperbolic functions written as trig functions or vice versa. Allow slips if they are recovered but clear and consistent errors score A0			
	Note: There may be other methods seen, e.g., using exponentials and "meeting in the middle"			
				(3)

(b)	e.g. $\frac{d}{dx}(\coth x) = -\text{cosech}^2 x$ or $\frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x}$ or $\frac{-\text{sech}^2 x}{\tanh^2 x}$ or $1-\coth^2 x$ etc. or e.g. $\frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2}$ or $\frac{2e^{2x}(e^{2x}-1) - 2e^{2x}(e^{2x}+1)}{(e^{2x}-1)^2}$ or $\frac{-4}{(e^x - e^{-x})^2}$ etc. Correct derivative of coth *x* in any form. Allow recovery if they write e.g. $-\text{cosec}^2 x$ when $-\text{cosech}^2 x$ is clearly implied by subsequent work.	B1
	e.g., $\text{sech}\, x - \text{cosech}^2 x = 0 \Rightarrow \text{sech}\, x = \text{cosech}^2 x \Rightarrow \frac{1}{\cosh x} = \frac{1}{\sinh^2 x} \Rightarrow$ $a\cosh^2 x + b\cosh x + c = 0$ or $a\,\text{sech}^2 x + b\,\text{sech}\, x + c = 0$ **or** $\text{sech}\, x - \text{cosech}^2 x = 0 \Rightarrow\Rightarrow \frac{2}{e^x + e^{-x}} - \left(\frac{2}{e^x - e^{-x}}\right)^2 = 0 \Rightarrow$ $\Rightarrow Ae^{4x} + Be^{3x} + Ce^{2x} + De^x + E = 0$ Sets $f'(x) = 0$ **and** uses correct identities to obtain a 3TQ in cosh *x* or sech *x* **or** substitutes the correct exponential forms and obtains a 5 term quartic in e^x	M1
	$\cosh^2 x - \cosh x - 1 = 0$ or $\text{sech}^2 x + \text{sech}\, x - 1 = 0$ oe **or** $\Rightarrow e^{4x} - 2e^{3x} - 2e^{2x} - 2e^x + 1 = 0$ oe Correct quadratic equation or correct quartic equation.	A1
	$\cosh x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \left(= \frac{1+\sqrt{5}}{2}\right)$ or e.g., $\left(\text{sech}\, x + \frac{1}{2}\right)^2 - \frac{1}{4} - 1 = 0 \Rightarrow \text{sech}\, x = \left(\frac{-1+\sqrt{5}}{2}\right)$ Solves quadratic resulting from sech *x* + their derivative of coth *x* = 0 Must obtain a **real and exact** value > 1 (or between 0 and 1 if sech used). Apply usual rules. (No need to reject invalid values) If no solving method seen one solution must be consistent with their equation. For the 5 term quartic in e^x progress is unlikely unless they proceed via e.g. $\left(e^{2x} - (1+\sqrt{5})e^x + 1\right)^2 = 0$	**d**M1
	$x = \text{arcosh}\left(\frac{1+\sqrt{5}}{2}\right) = \ln\left(\frac{1+\sqrt{5}}{2} + \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2 - 1}\right)$ **or** $\frac{e^x + e^{-x}}{2} = \frac{1+\sqrt{5}}{2} \Rightarrow e^{2x} - (1+\sqrt{5})e^x + 1 = 0 \Rightarrow e^x = \frac{1+\sqrt{5} + \sqrt{(1+\sqrt{5})^2 - 4}}{2} \Rightarrow x = ...$ Uses correct logarithmic form or exponentials to find *x* as a ln of an exact value. Exponential definition must be correct and quadratic solving subject to usual rules or consistent with their equation leading to a value of $e^x > 0$	**dd**M1
	$\Rightarrow x = \ln\left(\frac{1}{2}(1+\sqrt{5}) + \sqrt{\frac{1}{2}(1+\sqrt{5})}\right)$ or accept $x = \ln\left(\frac{1+\sqrt{5}}{2} + \sqrt{\frac{1+\sqrt{5}}{2}}\right)$ Note that $x = \ln\frac{1}{2}(1+\sqrt{5}) + \sqrt{\frac{1}{2}(1+\sqrt{5})}$ scores A0	A1
		(6)
		Total 9

Correct work in (b) leading to:

$$\cosh^2 x - \cosh x - 1 = 0 \Rightarrow \cosh x = \frac{1+\sqrt{5}}{2}$$

$$x = \text{arcosh}\left(\frac{1+\sqrt{5}}{2}\right) = \ln\left(\frac{1+\sqrt{5}}{2} + \sqrt{\frac{1+\sqrt{5}}{2}}\right)$$

With no evidence where the $\sqrt{\frac{1+\sqrt{5}}{2}}$ comes from, scores: B1M1A1dM1ddM0A0

Question Number	Scheme	Notes	Marks
8(a)	$\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{y}{4}$ or $2y\frac{\mathrm{d}y}{\mathrm{d}x}=8$ or $\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\frac{1}{2}\right)\left(2\sqrt{2}\right)x^{-\frac{1}{2}}$ or $\left(\frac{1}{2}\right)\left(2\sqrt{2}\right)\left(\frac{2\sqrt{2}}{y}\right)$ **oe** Any correct equation in $\frac{\mathrm{d}x}{\mathrm{d}y}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y or x		B1
	$\left(\int\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y=\right)\int\sqrt{1+\left(\frac{y}{4}\right)^2}(\mathrm{d}y)$ or $\left(\int\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,.\frac{\mathrm{d}x}{\mathrm{d}y}\mathrm{d}y=\right)\int\sqrt{1+\left(\frac{4}{y}\right)^2}\,.\frac{y}{4}(\mathrm{d}y)$ Forms $\int\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}(\mathrm{d}y)$ or $\int\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,.\frac{\mathrm{d}x}{\mathrm{d}y}(\mathrm{d}y)$ correctly with their derivative		M1
	$x=18 \Rightarrow y^2=144 \Rightarrow \beta=12,\ \alpha=24$ $\Rightarrow(\text{perimeter of } R=)\ 24+2\int_0^{12}\sqrt{1+\frac{y^2}{16}}\,\mathrm{d}y$	Correct expression	A1
			(3)

(b)	$y=4\sinh u \Rightarrow \frac{dy}{du}=4\cosh u$	Correct derivative. Condone $\frac{dy}{dx}=4\cosh u$	B1
	$\int\sqrt{1+\frac{y^2}{16}}\,dy=\int\sqrt{1+\frac{(4\sinh u)^2}{16}}(4\cosh u)(du)$ $\left(=4\int\cosh^2 u\,du\right)$	Full substitution, correct for their $\frac{dy}{du}$	M1
	$\int\cosh^2 u\,du=\int\left(\frac{1}{2}\cosh 2u+\frac{1}{2}\right)du=\frac{1}{4}\sinh 2u+\frac{1}{2}u$ **or** $\int\left(\frac{e^u+e^{-u}}{2}\right)^2du=\int\left(\frac{e^{2u}}{4}+\frac{1}{2}+\frac{e^{-2u}}{4}\right)du=\frac{e^{2u}}{8}+\frac{1}{2}u-\frac{e^{-2u}}{8}$ **d**M1: Uses $\cosh^2 u=\pm\frac{1}{2}\cosh 2u\pm\frac{1}{2}$ and integrates to obtain $a\sinh 2u+bu$ **or** uses $k\left(e^u+e^{-u}\right)$ for cosh u, expands and integrates to obtain $ae^{2u}+bu+ce^{-2u}$ A1: Correct integration		**d**M1 A1
	Perimeter of R:		**dd**M1
	$=24+(2)(4)\left[\frac{1}{4}\sinh 2u+\frac{1}{2}u\right]_0^{\text{arsinh }3=\ln(3+\sqrt{10})}$ $=24+2\left[2\sinh u\sqrt{1+\sinh^2 u}+2u\right]_0^{\text{arsinh}3=\ln(3+\sqrt{10})}$ $=24+2\left[(2)(3)\sqrt{1+3^2}+2\ln\left(3+\sqrt{10}\right)\right]$	$=24+(2)(4)\left[\frac{e^{2u}}{8}+\frac{1}{2}u-\frac{e^{-2u}}{8}\right]_0^{\ln(3+\sqrt{10})}$ $=24+e^{2\ln(3+\sqrt{10})}-e^{-2\ln(3+\sqrt{10})}+4\ln\left(3+\sqrt{10}\right)$ $24+\left(3+\sqrt{10}\right)^2-\frac{1}{\left(3+\sqrt{10}\right)^2}+4\ln\left(3+\sqrt{10}\right)$	
	Substitutes arsinh3 and/or $\ln\left(3+\sqrt{3^2+1}\right)$ into their expression using correct identities or correctly removes exponentials to obtain a numerical expression in constants and lns only Accept use of calculator here e.g. $\sinh(2\text{arsinh}3)=6\sqrt{10}$		
	$24+12\sqrt{10}+4\ln\left(3+\sqrt{10}\right)$ or, e.g., $4\left(6+3\sqrt{10}+\ln\left(3+\sqrt{10}\right)\right)$	Correct answer – any exact simplified equivalent	A1
	Note: Integration by calculator is likely to access the first two marks only		**(6)**
			Total 9
	TOTAL FOR PAPER: 75 MARKS		