Question	Scheme	Marks
1	$2\sinh^2 x + 3\cosh x = 7 \Rightarrow 2(\cosh^2 x - 1) + 3\cosh x = 7$	M1
	$\Rightarrow 2\cosh^2 x + 3\cosh x - 9 = 0$	A1
	$\Rightarrow (2\cosh x - 3)(\cosh x + 3) = 0 \Rightarrow \cosh x = \frac{3}{2}, \cancel{-3}$	M1
	$\Rightarrow x = \text{arcosh}\frac{3}{2} = (\pm)\ln\left(\frac{3}{2} + \sqrt{\frac{9}{4} - 1}\right)$	M1
	$x = \ln\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right), \ln\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)$	A1A1
		(6)
Alt	$2\sinh^2 x + 3\cosh x = 7 \Rightarrow 2\left(\frac{e^x - e^{-x}}{2}\right)^2 + 3\left(\frac{e^x + e^{-x}}{2}\right) = 7$	M1
	$\Rightarrow e^{4x} + 3e^{3x} - 16e^{2x} + 3e^x + 1 = 0$	A1
	$\Rightarrow (e^{2x} + 6e^x + 1)(e^{2x} - 3e^x + 1) = 0 \Rightarrow e^x = \ldots$	M1
	$e^x = \cancel{-3 \pm 2\sqrt{2}}, \frac{3}{2} \pm \frac{\sqrt{5}}{2} \Rightarrow x = \ln\left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right)$	M1
	$x = \ln\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right), \ln\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)$	A1A1
		(6)
		(6 marks)

Notes:

M1: Attempts to use the correct identity $\cosh^2 x - \sinh^2 x = 1$ to form an equation in cosh x only. (Allow slips rearranging a correctly stated identity.)
A1: Correct 3 term quadratic in cosh x. The "=0" may be implied by their attempt to solve.
M1: Solves the 3 term quadratic (usual rules) leading to at least one exact real value for cosh x (which may be negative). The negative value may not be seen, but need not be rejected at this stage. If using a calculator they must state at least one correct root for their equation.
M1: Forms a solution in terms of natural logarithms from a positive solution to their equation using the correct formula or via using the correct exponential form and solving a 3 term quadratic in e^x.
A1: One value correct.
A1: Both values correct, accepting equivalents, and no others. Equivalents are e.g.

$x = \pm\ln\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right), \quad \pm\ln\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)$ Condone missing brackets.

Alt
M1: Substitutes **correct** definitions for sinh x and cosh x in terms of exponentials.
A1: Correct simplified quartic in e^x.
M1: Solves their 5 term quartic as far as $e^x = \ldots.$ to achieve exact values (at least one). There should be a recognisable attempt at solving, e.g. via a product of two quadratics, with at least first and last terms

matching their quartic, but allow answers only from a calculator – but must be correct and exact for their equation.

M1: Proceeds to a solution in terms of natural logarithms from a positive solution to their equation using the correct formula. Not dependent, so this may be scored from decimal solutions to the quartic, for proceeding from $e^x = \ldots$ to $x = \ln\ldots$

A1: One value correct.

A1: Both values correct, accepting equivalents, and no others.

There may be other variations via substitutions etc.

M1: Attempts to use the correct identities to form an equation in one "variable".

A1: Correct simplified equation.

M1: Full attempt to solve the equation to at least $\cosh x = \ldots$ or $e^x = \ldots$ (exact)

M1: Forms a solution in terms of natural logarithms from a valid solution to their equation using the correct formula.

A1: One value correct.

A1: Both values correct, accepting equivalents, and no others. Equivalents are e.g.

$x = \pm\ln\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right), \ \pm\ln\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)$ Condone missing brackets.

Question	Scheme	Marks
2.(a)	$kx^2 - y^2 = 9$	
	$\frac{x^2}{9/k} - \frac{y^2}{9} = 1 \Rightarrow a^2 = \frac{9}{k}, b^2 = 9$ or Focus (6, 0) $\Rightarrow 6 = ae$	B1
	$\frac{x^2}{9/k} - \frac{y^2}{9} = 1 \Rightarrow a^2 = \frac{9}{k}, b^2 = 9$ and Focus (6, 0) $\Rightarrow 6 = ae$	M1
	E.g. $b^2 = a^2(e^2-1) \Rightarrow 9 = a^2\left(\frac{36}{a^2}-1\right) = 36 - a^2 \Rightarrow a^2 = ...(27) \Rightarrow k = ...$ or $b^2 = a^2(e^2-1) \Rightarrow 9 = \frac{36}{e^2}(e^2-1) \Rightarrow 9e^2 = 36e^2 - 36 \Rightarrow e^2 = ...\left(\frac{36}{27}\right) \Rightarrow k = ...$ $b^2 = a^2(e^2-1) \Rightarrow 9 = \frac{9}{k}\left(\frac{36k}{9}-1\right) = 36 - \frac{9}{k} \Rightarrow k = ...$	M1
	$\left(\Rightarrow \frac{9}{k} = 27\right) \Rightarrow k = \frac{1}{3}$	A1
		(4)
(b)	E.g. $a = "\sqrt{27}"$, directrix is $x = \pm\frac{a}{e} \Rightarrow (x=) \pm \frac{"\sqrt{27}"}{6/\sqrt{27}} = ...$	M1
	$x = \frac{9}{2}$	A1
		(2)
		(6 marks)

Notes

(a)

B1: Either Correctly identifies b^2 and a^2 (or b and a) in terms of k for the hyperbola (may be implied). (isw if they put $\pm$... square rooting.) May be implied by working. May be seen in (b)

Or Uses the focus to set up a correct equation relating a and e or k and e. Note that $a^2 + b^2\left(=(ae)^2\right) = 36$ is an alternative correct equation. May be implied by working.

M1: Both of the above correctly attempted or implied in the working but accept $a^2 = 9k$ or $\frac{k}{9}$ as an attempt (and similar for b^2).

M1: Full method to find k using $b^2 = a^2(e^2-1)$ with their $ae = 6$ and their a and b. E.g. uses the eccentricity equation with their b and with their a or e in terms of a, e, or k appropriately, to set up and solve an equation in one variable, reaching a value for k. Condone confusion between a and a^2 or b and b^2 when substituting into the formulae for this mark.

A1: Correct k

(b)

M1: Full method to find $(\pm)\frac{a}{e}$ using their a and e (finding these if not already seen) or k. They must be using their a correctly, M0 if they have correctly identified a^2 but use it incorrectly here. Allow if $\pm$ is used and/or the x is missing for this mark.

A1: Correct equation only from fully correct work, and must be an equation not just value. A0 if $\pm$ given, they must be choosing the positive to correspond to the focus.

Question	Scheme	Marks
3(i)	$\int \frac{1}{4x^2+12x+25}\,\mathrm{d}x$	
	$\left\{4x^2+12x+25=4\left(x^2+3x\right)+25\right\}$ $=4\left(x+\frac{3}{2}\right)^2+\ldots$ or $4x^2+12x+25=\left(2x+3\right)^2+\ldots$	M1
	$4\left(x+\frac{3}{2}\right)^2+16$ or $\left(2x+3\right)^2+16$	A1
	$\int \frac{1}{4x^2+12x+25}\,\mathrm{d}x=\frac{1}{4}\int \frac{1}{\left(x+\frac{3}{2}\right)^2+4}\,\mathrm{d}x=k\arctan\left(\frac{"x+\frac{3}{2}"}{"2"}\right)(+c)$ or $\int \frac{1}{4x^2+12x+25}\,\mathrm{d}x=\int \frac{1}{\left(2x+3\right)^2+4^2}\,\mathrm{d}x=k\arctan\left(\frac{"2x+3"}{"4"}\right)(+c)$	M1
	$=\frac{1}{8}\arctan\left(\frac{2x+3}{4}\right)(+c)$ (oe)	A1
		(4)
(ii)	$\int_3^9 \frac{1}{\sqrt{x^2+4x-17}}\,\mathrm{d}x=\ln a$	
	$x^2+4x-17=\left(x+2\right)^2-4-17$	B1
	$\int \frac{1}{\sqrt{x^2+4x-17}}\,\mathrm{d}x=\int \frac{1}{\sqrt{(x+2)^2-21}}\,\mathrm{d}x=\operatorname{arcosh}\left(\frac{x+2}{\sqrt{21}}\right)$ or $\ln\left(x+2+\sqrt{(x+2)^2-21}\right)$ or $\ln\left(\frac{x+2}{\sqrt{21}}+\sqrt{\left(\frac{x+2}{\sqrt{21}}\right)^2-1}\right)$ Alt: $u=x+2\Rightarrow\int\frac{1}{\sqrt{u^2-21}}\,\mathrm{d}u=\operatorname{arcosh}\left(\frac{u}{\sqrt{21}}\right)$ or $\ln\left(u+\sqrt{u^2-21}\right)$ etc	M1 A1
	$\int_3^9 \frac{1}{\sqrt{x^2+4x-17}}\,\mathrm{d}x=\operatorname{arcosh}\left(\frac{11}{\sqrt{21}}\right)-\operatorname{arcosh}\left(\frac{5}{\sqrt{21}}\right)$ or $\ln\left(11+\sqrt{11^2-21}\right)-\ln\left(5+\sqrt{5^2-21}\right)$	M1
	$\left(=\ln\left(\frac{11}{\sqrt{21}}+\sqrt{\frac{121}{21}-1}\right)-\ln\left(\frac{5}{\sqrt{21}}+\sqrt{\frac{25}{21}-1}\right)\right)=\ln\left(\frac{11+\sqrt{100}}{5+\sqrt{4}}\right)$	dM1
	$=\ln 3$	A1
		(6)
		(10 marks)

Notes:

(i)

M1: Factors out the 4 and completes the square achieving $4\left(x+\frac{3}{2}\right)^2+\ldots$ or without factoring the 4 achieves $(2x+3)^2+\ldots$ where … is anything other than 25. Condone for this mark $4\int\frac{1}{\left(x+\frac{3}{2}\right)^2+\ldots}\,\mathrm{d}x$ but this is A0 if no independent correct expression is shown.

A1: Correct completed square form.

M1: Attempts the integration with arctan used. Look for $k\arctan(l\times\text{their linear term})$. Allow with any *k* and *l*, including 1 or $\frac{1}{4}$. (May be via a substitution but must return to integral in *x* of the correct form for the mark.)

If an incorrect completion of the square results in $(ax+b)^2-c^2$ then this mark may be awarded for $\ldots\ln\left(\frac{ax+b-c}{ax+b+c}\right)$

A1: Correct answer, need not be fully simplified but must have at least combined multiple fractions, e.g. $\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$ must have been combined. Accept alternative forms of arctan, e.g. $\tan^{-1}$ or atan.

(ii)

B1: Correct completed square form for the quadratic.

M1: Attempts the integration to either arcosh or ln form. Form should be correct for their $x+2$ and a. Allow with ln| | and give benefit of doubt for the M if mod is required. They may use a substitution for this mark, $u=x+2$, and may be scored for the integral in terms of u. If a substitution such as $x+2=\cosh(u)$ or $x+2=\sqrt{21}\sec u$ is used they must carry out the full process – with supporting working shown - to reach an integral in u **with correct limits identified** as part of the method, or reversal of the substitution must carried out.

A1: Correct integral (may be in terms of u if the substitution $u=x+2$ is used), or correct integral **and** correct limits if a hyperbolic substitution is used. Ignore spurious statements of equality if they give both forms, and award for either correct form. Allow with moduli in the logs.

Note that $x+2=\sqrt{21}\sec u$ will lead to an integral $\left[\ln\left|\tan u+\sec u\right|\right]_{\text{arcsec}\frac{5}{\sqrt{21}}}^{\text{arcsec}\frac{11}{\sqrt{21}}}$

M1: Applies the limits 3 and 9 to their integral of a correct form (allowing a minor slip, e.g. a sign error inside a ln), or correct limits for their variable if they use a substitution, in an integral of the correct form (may be scored if M0A0 was awarded for insufficient evidence). Evidence of the application of limits must be seen – do not accept solutions which go directly to the answer. All steps are working are asked for in the question.

dM1: Simplifies to a single ln term, applying the arcosh ln (allowing a minor slip, e.g. a sign error inside a ln) formula first if necessary. Again, there must be sufficient evidence after substitution before the final simplified answer.

A1: cso

Question	Scheme	Marks
4(a)	$\begin{pmatrix}4&0&2\\0&4&a\\2&a&20/3\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}10\\\ldots\\\ldots\end{pmatrix}$	M1
	$\lambda_1=10$	A1
		(2)
(b)	$\begin{pmatrix}4&0&2\\0&4&a\\2&a&20/3\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}10\\20\\30\end{pmatrix}\Rightarrow 8+3a=20 \text{ or } 2+2a+\frac{20}{3}\times3=30$	M1
	$\Rightarrow a=4*$	A1*
		(2)
(c)	$\begin{vmatrix}4-\lambda&0&2\\0&4-\lambda&4\\2&4&\frac{20}{3}-\lambda\end{vmatrix}=(4-\lambda)\left((4-\lambda)\left(\frac{20}{3}-\lambda\right)-16\right)+2\left(0-2(4-\lambda)\right)$	M1
	$\left[-\lambda^3+\frac{44}{3}\lambda^2-\frac{148}{3}\lambda+\frac{80}{3}=0\right]\Rightarrow\frac{1}{3}(4-\lambda)(3\lambda^2-32\lambda+20)=0\Rightarrow\lambda=\ldots,\ldots$	dM1
	$\lambda=4,\frac{2}{3}(,10)$	A1
		(3)
(d)	$\begin{pmatrix}4&0&2\\0&4&4\\2&4&20/3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\lambda\begin{pmatrix}x\\y\\z\end{pmatrix}\Rightarrow\begin{cases}(4-\lambda)x+2z=0\\(4-\lambda)y+4z=0\\2x+4y+\left(\frac{20}{3}-\lambda\right)z=0\end{cases}\Rightarrow\ldots$	M1
	$\lambda=\frac{2}{3}\Rightarrow k\begin{pmatrix}3\\6\\-5\end{pmatrix}$ or $\lambda=4\Rightarrow m\begin{pmatrix}2\\-1\\0\end{pmatrix}$	A1
	Normalised $\Rightarrow\frac{v}{\lVert v\rVert}=\ldots\ \lambda=\frac{2}{3}\Rightarrow\frac{1}{\sqrt{70}}\begin{pmatrix}3\\6\\-5\end{pmatrix}$ and $\lambda=4\Rightarrow\frac{1}{\sqrt5}\begin{pmatrix}2\\-1\\0\end{pmatrix}$; $\lambda=10\Rightarrow\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$	M1 A1;B1
		(5)
(e)	$\mathbf{P}=\begin{pmatrix}1/\sqrt{14}&2/\sqrt5&3/\sqrt{70}\\2/\sqrt{14}&-1/\sqrt5&6/\sqrt{70}\\3/\sqrt{14}&0&-5/\sqrt{70}\end{pmatrix},\ \mathbf{D}=\begin{pmatrix}10&0&0\\0&4&0\\0&0&2/3\end{pmatrix}$	B1ft B1ft
		(2)
		(14 marks)

Notes:
(a) **M1:** Correct attempt to find the eigenvalue. Need only see top row calculated. May come from attempts at using $(\mathbf{M}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ **A1:** Correct value. **(b)** **M1:** Sets up a correct equation in a for their eigenvalue in (a). Must be extracted from the matrix equation. May come from attempts at using $(\mathbf{M}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$. Condone any unused incorrect equations. **A1*:** Shows that a = 4 from correct work. **(c)** **M1:** Attempts to expand $\det(\mathbf{M}-\lambda\mathbf{I})$. This may be along any row or down any column, or via a "shoelace" approach. Condone sign slips, but the overall structure should be correct. Working must be shown – do not accept solutions that just state the factorised form. The expansion must be seen in some form. Accept alternative approaches. **dM1:** Sets the characteristic polynomial equal to zero (which may be implied) and solves by valid method (may be by calculator) – dependent so as a minimum an attempt at the expansion must have been seen but the cubic need not have been fully simplified. **A1:** Correct remaining eigenvalues. **(d) Note – you may allow recovery of the final 3 marks in this part if the relevant normalisation work is completed in part (e).** **M1:** Sets up the eigenvector equations in general terms or for one of their eigenvalues and makes some attempt to use them (allow even for the given eigenvector for this mark). Working must be shown, M0 if correct vectors are stated with no supporting working. **A1:** Either of the two remaining eigenvectors found, need not be normalised for this mark. Must be given as a vector, but accept row vectors rather than column vectors. **M1:** Applies a correct process to normalise at least one of their vectors. This may be done on the given eigenvector (working need not be shown for this one). **A1:** Both normalised eigenvectors for $\lambda = 4, \frac{2}{3}$ correct. (Row or column vectors acceptable.) **B1:** For correct normalised vector for eigenvalue 10. (Row or column vector acceptable.) You may see $\lambda=\frac{2}{3} \Rightarrow \begin{pmatrix} 3\sqrt{70}/70 \\ 3\sqrt{70}/35 \\ -\sqrt{70}/14 \end{pmatrix}$ and $\lambda=4 \Rightarrow \begin{pmatrix} 2\sqrt{5}/5 \\ -\sqrt{5}/5 \\ 0 \end{pmatrix}$; $\lambda=10 \Rightarrow \begin{pmatrix} \sqrt{14}/14 \\ \sqrt{14}/7 \\ 3\sqrt{14}/14 \end{pmatrix}$ or other equivalent versions of these vectors, e.g $\lambda=\frac{2}{3} \Rightarrow \frac{\sqrt{70}}{14}\begin{pmatrix} 3/5 \\ 6/5 \\ -1 \end{pmatrix}$ **(e)** **B1ft:** For either a correct **P** for their **normalised** eigenvectors providing they are non-zero **or** for a correct **D** for their eigenvalues. **B1ft:** For both a correct matching **P** and **D** for their **normalised** eigenvectors and their eigenvalues, providing there is no zero column or row. Note: If **D** and **P** are not labelled then treat their diagonal matrix as **D** and other matrix as **P**. If they are incorrectly labelled (ie **D** as **P**) then withhold the second B (if earned).

Question	Scheme	Marks
5(a)	$x=(2t+3)^{\frac{3}{2}} \qquad y=\frac{3}{2}t^2+3t+6 \qquad -\frac{3}{2}\leqslant t\leqslant 3$	
	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{3}{2}(2t+3)^{\frac{1}{2}}\times 2 \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=3t+3$	M1A1
	$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2=9(2t+3)+(3t+3)^2=\ldots$	M1
	$\left[=9(t^2+4t+4)\right]=9(t+2)^2$	A1
		(4)
(b)	Arc length $=\int_{-\frac{3}{2}}^{3}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t=3\int_{-\frac{3}{2}}^{3}t+2\,\mathrm{d}t=3\left[\frac{t^2}{2}+2t\right]_{-\frac{3}{2}}^{3}$	M1
	$=3\left[\left(\frac{9}{2}+6\right)-\left(\frac{9}{8}-3\right)\right]=\ldots$	M1
	$=\frac{297}{8}$	A1
		(3)
(c)	Surface area $=2\pi\int_{-\frac{3}{2}}^{3}y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}(\mathrm{d}t)=6\pi\int_{-\frac{3}{2}}^{3}\left(\frac{3}{2}t^2+3t+6\right)(t+2)(\,\mathrm{d}t)$	B1ft
	$=6\pi\int_{-\frac{3}{2}}^{3}\frac{3}{2}t^3+6t^2+12t+12\,\mathrm{d}t=6\pi\left[\frac{3}{8}t^4+2t^3+6t^2+12t\right]_{-\frac{3}{2}}^{3}$	M1
	$=6\pi\left[\left(\frac{3}{8}3^4+2\times 3^3+6\times 3^2+12\times 3\right)-\left(\frac{3}{8}\left(\frac{81}{16}\right)+2\left(-\frac{27}{8}\right)+6\left(\frac{9}{4}\right)+12\left(-\frac{3}{2}\right)\right)\right]$ $=6\pi\left(\frac{1395}{8}-\frac{-1197}{128}\right)=\ldots$	dM1
	$=\frac{70551\pi}{64}$	A1
		(4)
		(11 marks)

Notes

(a)

M1: Attempts both derivatives with correct form for each, $\frac{\mathrm{d}x}{\mathrm{d}t}=A(2t+3)^{\frac{1}{2}}$, $\frac{\mathrm{d}y}{\mathrm{d}t}=Bt+3$.

A1: Both correct.
M1: Squares and adds their two derivatives and proceeds to simplify to a 3TQ
A1: Correct answer obtained with no incorrect working seen. Allow $a=9$ stated as answer if they reach at least $9(t^2+4t+4)$ but do not show the factorisation.

(b)
M1: Applies the arc length formula and integrates the square root of their answer to (a) achieving a quadratic answer. (Allow if their *a* is not square rooted.) Limits not needed for this mark. Note they may integrate via reverse chain rule to $A(t+2)^2$ (which is equivalent).

M1: Applies the correct limits to their integral, which must have been from an attempt at the correct formula, but allow if $a(t+2)^2$ was used following a stated correct formula. Must have seen an increase in power in at least one term in the linear function – or quadratic if $a(t+2)^2$ was used per note above. Do not be concerned about checking if they just give values for each bracket (or indeed just an answer) – assume the have applied the correct limits unless there is contrary evidence (e.g. wrong limits stated or seen applied).
A1: Correct answer.
(c)
B1ft: Applies the surface area formula correctly with their answer to (a). The 2π and correct limits must be seen at some stage. The d*t* may be implied.
M1: Expands and integrates from a correct attempt at the area formula. Must be using $\int y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,(dt)$ but again allow here if the correct formula is quoted but they subsequently forget to take the square root of (a). May be missing limits and 2π for this mark.
dM1: Applies the correct limits and simplifies to a single term. Do not be concerned about checking if they just give values for each bracket (or indeed just an answer) – assume the have applied the correct limits unless there is contrary evidence (e.g. wrong limits stated or seen applied).
A1: Correct answer, but do not isw if they try and add "ends" to the surface and award A0 in such cases.

Note: If they neglect to square root the answer in both parts the second M in (b) and both M's in (c) may be gained. This will require a cubic integral in (b) and integrating a quartic to quintic in (c).

Question	Scheme	Marks
6(a)	$\begin{pmatrix}2\\0\\7\end{pmatrix}\times\begin{pmatrix}3\\-2\\3\end{pmatrix}=\begin{pmatrix}0-(-14)\\-(6-21)\\-4-0\end{pmatrix};=\begin{pmatrix}14\\15\\-4\end{pmatrix}$	M1; A1
		(2)
(b)	Direction of l is $\pm\left(\begin{pmatrix}1\\7\\3\end{pmatrix}-\begin{pmatrix}-1\\5\\2\end{pmatrix}\right)=\pm\begin{pmatrix}2\\2\\1\end{pmatrix}$	B1
	Angle between line and normal to plane is given by $\left(\cos\beta=\right)\dfrac{14\times2+15\times2-4\times1}{\sqrt{437}\sqrt{9}}\left(=\dfrac{18}{\sqrt{437}}\right)$	M1
	So angle between plane and line is $\alpha=90°-\beta=...$	M1
	$=59°$	A1
		(4)
(c)	$\left\{\text{Equation of plane is } \mathbf{r}.\begin{pmatrix}14\\15\\-4\end{pmatrix}=\right\}\begin{pmatrix}-1\\5\\2\end{pmatrix}.\begin{pmatrix}14\\15\\-4\end{pmatrix}=53$	M1 A1
	So perpendicular distance is $\dfrac{\lvert 14\times1+15\times7-4\times3-53\rvert}{\sqrt{437}}$	M1
	$=\dfrac{54}{\sqrt{437}}$	A1
		(4)
		(10 marks)

Notes

(a)

M1: Attempts the vector product of the two directions of the plane. If no method shown, at least two correct coordinates implies the method. Alternatively, sets up and solves a pair of simultaneous equations using dot products of a general vector with the directions. E.g.

$$\left.\begin{array}{r}(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\bullet(2\mathbf{i}+7\mathbf{k})=0\\(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\bullet(3\mathbf{i}-2\mathbf{j}+3\mathbf{k})=0\end{array}\right\}\Rightarrow\begin{array}{c}2a+7c=0\\3a-2b+3c=0\end{array}\Rightarrow a=..c,b=..c$$

A1: Correct normal to the plane.

(b)

B1: Correct direction vector for l, either direction.

M1: Attempts a relevant dot product of the direction of l and normal to the plane. If no method shown the value given must be correct for their vectors. Need not be named correctly, so accept if they think the angle between these is what they need for this mark.

M1: Correct method to find the required angle. Must be using the correct vectors, but may be scored if the previous M is withheld for incorrect method for the scalar product if they have stated this is what they were attempting to find. Look for use of cos β and subtracting from 90º, or for use of sin α with the dot product.

A1: Correct final angle (awrt). You may isw incorrect subsequent rounding following a correct answer.

(c)

M1: Attempts to find "$\pm d$" for the equation of the plane with their normal from (a) and the point B.

A1: Correct value or equation for plane.

M1: Attempts the perpendicular distance formula or any other full method for the distance, allowing minor slips (e.g. confusion in sign of d). E.g. $\frac{\left|\overrightarrow{OA}\bullet\mathbf{n}-\overrightarrow{OB}\bullet\mathbf{n}\right|}{|\mathbf{n}|}=\ldots$ is an equivalent method.

A1: Correct answer.

Alt (c)	$\lvert AB\rvert=\left\lvert\begin{pmatrix}2\\2\\1\end{pmatrix}\right\rvert=\sqrt{4+4+1}=3$	**M1** **A1**
	So perpendicular distance is $\lvert AB\rvert\sin\alpha=\lvert AB\rvert\cos(90-\alpha)=\lvert AB\rvert\cos\beta=3\times"\frac{18}{\sqrt{437}}"$	**M1**
	$=\frac{54}{\sqrt{437}}$	**A1**
		(4)

M1: Correct method for the distance AB

A1: Correct AB.

M1: Attempts correct trig ratio with the exact answer for the relevant cosine from part (b).

A1: Correct answer.

Alt 2 (c)	$\hat{\mathbf{n}}=\frac{\mathbf{n}}{\lvert\mathbf{n}\rvert}=\frac{14\mathbf{i}+15\mathbf{j}-4\mathbf{k}}{\sqrt{437}}$ or $\overrightarrow{AB}\bullet\mathbf{n}=(2\mathbf{i}+2\mathbf{j}+\mathbf{k})(14\mathbf{i}+15\mathbf{j}-4\mathbf{k})=54$	**M1** **A1**
	So perpendicular distance is $\left\lvert\overrightarrow{AB}\bullet\hat{\mathbf{n}}\right\rvert=\left\lvert\frac{(2\mathbf{i}+2\mathbf{j}+\mathbf{k})(14\mathbf{i}+15\mathbf{j}-4\mathbf{k})}{\sqrt{437}}\right\rvert=\ldots$	**M1**
	$=\frac{54}{\sqrt{437}}$	**A1**
		(4)

M1: Attempts the unit normal vector or correct at $\overrightarrow{AB}\bullet\mathbf{n}$.

A1: Correct vector or correct value

M1: Applies formula for distance between planes.

A1: Correct answer.

Alt 3 (c)	$\left\{\text{Equation of plane is } \mathbf{r}.\begin{pmatrix}14\\15\\-4\end{pmatrix}=\right\}\begin{pmatrix}-1\\5\\2\end{pmatrix}.\begin{pmatrix}14\\15\\-4\end{pmatrix}=53$	**M1** **A1**
	Let X be where shortest distance to plane occurs. Then $\overrightarrow{AX}=\mathbf{i}+7\mathbf{j}+3\mathbf{k}-t\left(14\mathbf{i}+15\mathbf{j}-4\mathbf{k}\right)$ So X is where $\left((1+14t)\mathbf{i}+(7+15t)\mathbf{j}+(3-4t)\mathbf{k}\right)\bullet\left(14\mathbf{i}+15\mathbf{j}-4\mathbf{k}\right)=53$ $\Rightarrow 437t+107=53\Rightarrow t=-\frac{54}{437}$ $\overrightarrow{AX}=-\frac{54}{437}\left(14\mathbf{i}+15\mathbf{j}-4\mathbf{k}\right)\Rightarrow\left\lvert\overrightarrow{AX}\right\rvert=\frac{54}{437}\times\sqrt{437}=\ldots$	**M1**
	$=\frac{54}{\sqrt{437}}$	**A1**
		(4)

M1: Attempts to find "*d*" for the equation of the plane with their normal from (a) and the point *B*.
A1: Correct value / equation of plane implied.
M1: Sets up equation for line through *A* and perpendicular to the plane and forms and solves with the equation of the plane to find the parameter for which this line intersects the plane and attempts the length of *AX*.
A1: Correct answer.

Question	Scheme	Marks
7(a)	$\frac{dy}{dx}=\frac{b\cos\theta}{-a\sin\theta}$ or $\frac{dy}{dx}=-\frac{2x}{a^2}\times\frac{b^2}{2y}$	M1
	$m_N=\frac{a\sin\theta}{b\cos\theta}$	A1
	Equation is $y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta}(x-a\cos\theta)$	M1
	$\Rightarrow by-b^2\sin\theta=ax\tan\theta-a^2\sin\theta\Rightarrow by=ax\tan\theta+(b^2-a^2)\sin\theta$ *	A1*
		(4)
(b)	B is $(0,-b)\Rightarrow -b^2=(b^2-a^2)\sin\theta$	B1
	Area $OBP=\frac{3}{4}b^2\Rightarrow\frac{1}{2}b\times a\cos\theta=\frac{3}{4}b^2$	M1
	$\Rightarrow 2a\cos\theta=3b\Rightarrow b^2=\frac{4}{9}\cancel{a^2}\cos^2\theta=\left(\cancel{a^2}-\frac{4}{9}\cancel{a^2}\cos^2\theta\right)\sin\theta$	dM1
	$\Rightarrow 4(1-\sin^2\theta)=(9-4(1-\sin^2\theta))\sin\theta$ $\Rightarrow 4\sin^3\theta+4\sin^2\theta+5\sin\theta-4=0$	ddM1
	$\Rightarrow(2\sin\theta-1)(2\sin^2\theta+3\sin\theta+4)=0\Rightarrow\sin\theta=\frac{1}{2}$ *	A1*
		(5)
(c)	$b=\frac{4}{3}\times\frac{1}{2}a\times\sqrt{1-\left(\frac{1}{2}\right)^2}=...\left(=\frac{\sqrt{3}}{3}a\right)$	M1
	P is $\left(\frac{\sqrt{3}}{2}a,\frac{\sqrt{3}}{6}a\right)$	B1 A1
		(3)
		(12 marks)

Notes:

(a)

M1: A correct method for $\frac{dy}{dx}$ or $-\frac{dx}{dy}$ in terms of θ (from parametric differentiation) or in terms of x and y (from implicit or Cartesian approaches).

A1: Correct gradient for the normal in terms of θ

M1: Correct method for the normal through P applied. There must have been a correct attempt at the gradient of the normal (ie negative reciprocal of their $\frac{dy}{dx}$, or directly finding $-\frac{dx}{dy}$). Allow recovery from gradients in terms of x and y only if correct substitution for them occurs. If using $y = mx + c$ they must proceed as far as finding c.

A1*: Correct completion to the equation given with no errors seen in working and an intermediate step (e.g. if using $y = mx + c$ they must substitute c back in before proceeding to the given answer).

(b)

B1: States or identifies the point B is $(0,-b)$ and substitutes into the normal equation to obtain the correct equation shown or equivalent.

M1: Correct attempt at the area and sets equal to $\frac{3}{4}b^2$ to form a second equation. Note there are other possible methods for the area of the triangle e.g. "shoelace"/determinant method, or use of separate triangles. Some will essentially give the same result as in the scheme, but others will give more complicated formulae, making the next M much more challenging (and unlikely to be gained).

dM1: Solves the equations simultaneously to achieve an equation in θ only. Alternatively, eliminates $\sin\theta$ and forms a polynomial equation in a^2 and b^2. This latter is unlikely to make further progress.

ddM1: Rearranges to an expanded (but not necessarily fully simplified) cubic in $\sin\theta$. For equivalent working in a^2 or b^2 they must proceed as far as finding a relationship between a^2 and b^2.

A1*: Fully correct work, showing the factorised cubic to justify the given value of sin θ. No need to show there is only one solution (we allow the factorised to imply the quadratic factor does not reduce). Note substitution of $\sin\theta = \frac{1}{2}$ into the equation is A0 as it provides no evidence of no other solutions.

For working in a and b the factorisation to deduce the relationship between a^2 and b^2 must have been seen.

(c)

M1: Correct process to find b in terms of a

B1: x coordinate correct

A1: y coordinate correct, accept $\frac{a}{2\sqrt{3}}$.

(b) Working in a and b:

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \frac{b^4}{\left(b^2 - a^2\right)^2} + \frac{9b^2}{4a^2} = 1 \Rightarrow 4a^2b^4 + 9b^2\left(b^2 - a^2\right)^2 = 4a^2\left(b^2 - a^2\right)^2$ dM1

$\Rightarrow 9b^6 - 18a^2b^4 + 17a^4b^2 - 4a^6 = 0 \Rightarrow \left(3b^2 - a^2\right)\left(3b^4 - 5a^2b^2 + 4a^4\right) = 0$

$\Rightarrow a^2 = 3b^2$ ddM1

$\Rightarrow \sin\theta = -\frac{b^2}{b^2 - 3b^2} = -\frac{1}{-2} = \frac{1}{2}$ A1

Note: A lot of solutions are not using that the coordinate of B is $(0,-b)$ but instead use $\sin\theta = \frac{1}{2}$ to show $\sin\theta = \frac{1}{2}$. These will score at most B0M1M0M0A0. as they have no equations to solve simultaneously.

Question	Scheme	Marks
8	$y = e^{3x}\cosh 2x$ (See notes for Alt via $n = 0$ for M1A1) $\Rightarrow \frac{dy}{dx} = 3e^{3x}\cosh 2x + e^{3x} \times 2\sinh 2x$ or $e^{3x}(3\cosh 2x + 2\sinh 2x)$	M1
	$= e^{3x}\left(\frac{5+1}{2}\cosh 2x + \frac{5-1}{2}\sinh 2x\right)$ (so the result is true for $n = 1$)	A1
	(Assume the result is true for $n = k$, then) $\frac{d^{k+1}y}{dx^{k+1}} = 3e^{3x}\left(\frac{5^k+1}{2}\cosh 2x + \frac{5^k-1}{2}\sinh 2x\right) + e^{3x}\left(\left(5^k+1\right)\sinh 2x + \left(5^k-1\right)\cosh 2x\right)$	M1
	$= \left(3\times\frac{5^k+1}{2} + 5^k - 1\right)e^{3x}\cosh 2x + \left(3\times\frac{5^k-1}{2} + 5^k + 1\right)e^{3x}\sinh 2x$ $= e^{3x}\left(\frac{5\times 5^k+1}{2}\cosh 2x + \frac{5\times 5^k-1}{2}\sinh 2x\right)$	dM1
	$= e^{3x}\left(\frac{5^{k+1}+1}{2}\cosh 2x + \frac{5^{k+1}-1}{2}\sinh 2x\right)$	A1
	Hence the result is also true for $n = k + 1$, so <u>if true for $n = k$ then true for $n = k + 1$</u>, and as also <u>true for $n = 1$</u>, so the result is <u>true for all positive integers</u>.	A1
		(6)
		(6 marks)

Notes:

Note: Condone use of n instead of k throughout the inductive step. If reasoned correctly full marks are available. Also note that going from $n = k - 1$ to $n = k$ is a correct method.

Note: Apply benefit of the doubt if poor handwriting makes it difficult to distinguish between 5^k and $5k$ in some places (ie powers written too lower) if they are treated correctly in subsequent steps. This will not be counted as errors affecting the A's.

M1: Attempts the first derivative - form must be correct, but coefficients may be incorrect. Alternatively, allow for substitution of $n = 0$ into the right hand side – accept as minimum seeing $e^{3x}\left(\frac{1+1}{2}\cosh 2x + \frac{1-1}{2}\sinh 2x\right)$.

A1: Correct derivative and reaches appropriate form, either by factoring out the e^{3x} or expanding the RHS of result for $n = 1$ and comparing, to be able to deduce the result is true for $n = 1$ (which may be tacit for this mark). If expanding the RHS the substitution of the $n = 1$ should be clear.

Either way a minimum of $e^{3x}\left(\frac{5+1}{2}\cosh 2x + \frac{5-1}{2}\sinh 2x\right)$ must be seen in their working in order to award this mark.

Alternatively, substituting $n = 0$, proceeds from the minimum shown to reach $e^{3x}\cosh 2x = y$ to establish true for $n = 0$.

M1: (Makes the inductive assumption and) attempts the $(k + 1)$-th derivative from the k-th derivative (but see first note above). Allow slips in coefficients.

dM1: Gathers the sinh $2x$ and cosh $2x$ terms and factors out these terms to unsimplified coefficients. The e^{3x} may or may not be factored out, or missing, or incorrect at this stage. Accept either form shown (or equivalent unsimplified coefficients) or similar with the outer bracket expanded. There must be a step with unsimplified gathered coefficients between the differentiation and final line shown in the scheme to score this mark. **A1:** Reaches the correct form from correct work. The e^{3x} must be factored out, or expansion of the RHS to demonstrate equivalents seen. Must have the "k+1" showing. Depends on the previous two method marks. **A1:** Makes appropriate concluding sentence covering the points indicated in scheme at some stage in the proof. Depends on all method marks having been scored and all work being correct. Must have reached at least the second line shown in the dM mark, and made an attempt at checking n =1 (though the first A mark need not have been scored if insufficient detail shown). Note M1A0M1dM1A0A1 is possible if the previous A marks have only been withheld due to the lack of sufficient detail shown but the working and intent is clear (e.g only reaching the final line of the dM box).		
8 Alt via exponentials	$y=e^{3x}\cosh 2x=\dfrac{e^{5x}+e^{x}}{2} \quad \Rightarrow \dfrac{dy}{dx}=\dfrac{5e^{5x}+e^{x}}{2}$	**M1**
	$=e^{3x}\left(\dfrac{5e^{2x}+e^{-2x}}{2}\right)=e^{3x}\left(\dfrac{(5+1)\left(e^{2x}+e^{-2x}\right)+(5-1)\left(e^{2x}-e^{-2x}\right)}{4}\right)$ $=e^{3x}\left(\dfrac{5+1}{2}\cosh 2x+\dfrac{5-1}{2}\sinh 2x\right)$ so the result is true for n = 1	**A1**
	(Assume the result is true for $n = k$, so $\dfrac{d^k y}{dx^k}=e^{3x}\left(\dfrac{5^k+1}{2}\dfrac{e^{2x}+e^{-2x}}{2}+\dfrac{5^k-1}{2}\dfrac{e^{2x}-e^{-2x}}{2}\right)=\dfrac{5^k}{2}e^{5x}+\dfrac{1}{2}e^{x}$ then) $\dfrac{d^{k+1}y}{dx^{k+1}}=\dfrac{5^k}{2}\times 5e^{5x}+\dfrac{1}{2}e^{x}$	**M1**
	$=\dfrac{5^{k+1}}{2}e^{5x}+\dfrac{1}{2}e^{x}=\dfrac{e^{3x}}{4}\left(2\times 5^{k+1}e^{2x}+2e^{-2x}\right)$ $=\dfrac{e^{3x}}{4}\left(5^{k+1}e^{2x}+5^{k+1}e^{-2x}+5^{k+1}e^{2x}-5^{k+1}e^{-2x}+e^{2x}+e^{-2x}+e^{-2x}-e^{2x}\right)$ $=\dfrac{e^{3x}}{2}\left(5^{k+1}\cosh 2x+5^{k+1}\sinh 2x+\cosh 2x-\sinh 2x\right)$	**dM1**
	$=e^{3x}\left(\dfrac{5^{k+1}+1}{2}\cosh 2x+\dfrac{5^{k+1}-1}{2}\sinh 2x\right)$	**A1**
	Hence the result is also true for $n = k$ +1, so <u>if true for $n = k$ then true for $n = k + 1$</u>, and as also <u>true for n = 1</u>, so the result is <u>true for all positive integers</u>.	**A1**
		(6)
		(6 marks)
Notes:		

M1: Replaces by the correct exponential forms and differentiates - form must be correct, but coefficients may be incorrect.
A1: Correct derivative and reaches appropriate form to deduce the result is true for $n = 1$ (which may be tacit for this mark). If expanding the RHS the substitution of the $n = 1$ should be clear. The same minimum as main scheme is expected except that the cosh $2x$ and sinh $2x$ may have been replaced first.
M1: (Makes the inductive assumption and) attempts the (k +1)-th derivative from the k-th derivative. Allow slips in coefficients.
dM1: Attempts to introduce the extra terms to establish the cosh $2x$ and sinh $2x$ in the expression. Allow slips, but the intention must be clear. Alternatively, considers the right hand side for $k + 1$, substitutes exponential forms and expands and simplifies.
A1: Fully correct work to establish the inductive step. The final expression must be reached (with the intermediate dM1 gained) from correct work, or have been used to work back to matching expressions.
A1: Makes appropriate concluding sentence covering the points indicated in scheme at some stage in the proof. Depends on all method marks having been scored and all work being correct. Must have reached at least the final line shown in the dM mark or equivalent work in reverse, and made an attempt at checking n =1 (though the first A mark need not have been scored if insufficient detail shown).
Note M1A0M1dM1A0A1 is possible if the previous A marks have only been withheld due to the lack of sufficient detail shown but the working and intent is clear (e.g only reaching the final line of the dM box).

Note: Some attempts may try to use integration to work backwards from $\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}$ to $\frac{\mathrm{d}^k y}{\mathrm{d}x^k}$. Such methods may use hyperbolics or exponentials and the scheme will apply similarly.
M1A1: as per main scheme.
M1: Considers result for $n = k + 1$ and attempts to integrate twice by parts, achieving correct form.

Let $A = \frac{5^{k+1}+1}{2}\cosh 2x + \frac{5^{k+1}-1}{2}\sinh 2x, \quad B = \left(5^{k+1}+1\right)\sinh 2x + \left(5^{k+1}-1\right)\cosh 2x$

$$I = \int \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}\mathrm{d}x = \frac{1}{3}\mathrm{e}^{3x}(A) - \frac{1}{3}\int \mathrm{e}^{3x}B\,\mathrm{d}x = \frac{1}{3}\mathrm{e}^{3x}(A) - \frac{1}{9}\mathrm{e}^{3x}B + \frac{4}{9}\int \mathrm{e}^{3x}A\,\mathrm{d}x$$

Or in exponentials

$$\int \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}\mathrm{d}x = \int\left(\frac{5^{k+1}+1}{2}\frac{\mathrm{e}^{5x}+\mathrm{e}^{x}}{2} + \frac{5^{k+1}-1}{2}\frac{\mathrm{e}^{5x}-\mathrm{e}^{x}}{2}\right)\mathrm{d}x = \frac{5^{k+1}+1}{2}\frac{\frac{1}{5}\mathrm{e}^{5x}+\mathrm{e}^{x}}{2} + \frac{5^{k+1}-1}{2}\frac{\frac{1}{5}\mathrm{e}^{5x}-\mathrm{e}^{x}}{2}$$

dM1: Makes I the subject

$$\Rightarrow 9I = 3\mathrm{e}^{3x}(A) - \mathrm{e}^{3x}B + 4I \Rightarrow I = \frac{1}{5}\mathrm{e}^{3x}(3A - B)$$

Or: $= \frac{\mathrm{e}^{3x}}{5}\left(\frac{5^{k+1}+1}{2}\frac{\mathrm{e}^{2x}+5\mathrm{e}^{-2x}}{2} + \frac{5^{k+1}-1}{2}\frac{\mathrm{e}^{2x}-5\mathrm{e}^{-2x}}{2}\right) = \frac{\mathrm{e}^{3x}}{5}\left(\frac{2\times 5^{k+1}\mathrm{e}^{2x}+10\mathrm{e}^{-2x}}{4}\right)$ leading to identifying

....(reverse work of exponentials in main scheme)

exponential forms for the hyperbolics.
A1: Achieves the correct result and gives due consideration to the constant of integration.

$$\Rightarrow \frac{\mathrm{d}^k y}{\mathrm{d}x^k} = I = \frac{1}{5}\mathrm{e}^{3x}\left(\frac{5^{k+1}+5}{2}\cosh 2x + \frac{5^{k+1}-5}{2}\sinh 2x\right) = \mathrm{e}^{3x}\left(\frac{5^k+1}{2}\cosh 2x + \frac{5^k-1}{2}\sinh 2x\right)$$

A1: Per main scheme (except allow if constant of integration is not considered).
If you are unsure in a particular case use the review system.