Question Number	Scheme	Notes	Marks
1(a)	$ae=\frac{13}{2}$ **or** $\frac{a}{e}=\frac{72}{13}$	One correct equation in a and e. Allow equivalent correct equations. Could include – or ± signs	B1
	e.g., $a=\frac{72}{13}e \Rightarrow \frac{72}{13}e^2=\frac{13}{2} \Rightarrow e^2=\ldots \left(\frac{169}{144}\right)$ or $a=\frac{13}{2e} \Rightarrow \frac{13}{2e^2}=\frac{72}{13} \Rightarrow e^2=\ldots \left(\frac{169}{144}\right)$	Having obtained two equations in a and e of the correct form i.e., $ae=p$ and $\frac{a}{e}=q$ $p,q\neq 0$, solves simultaneously to find a <u>positive</u> value for e^2 (no requirement for e > 1) or e. Condone poor algebra provided a value is obtained. May find a first.	M1
	$e=\frac{13}{12}$ or $1\frac{1}{12}$ or $1.08\dot{3}$. Not $\pm\frac{13}{12}$ unless negative value clearly rejected in this part.		A1
			(3)
(b)	$\left\{a=\frac{72}{13}\times\frac{13}{12}=6 \text{ or } a=\frac{13}{2\left(\frac{13}{12}\right)}=6\right\}$ $b^2=a^2\left(e^2-1\right)=\ldots$ $\left\{b^2=6^2\left(\left(\frac{13}{12}\right)^2-1\right)=\frac{25}{4} \text{ or } b=\frac{5}{2}\right\}$	With any value for a, which might be seen in part (a), and their e, **uses** a correct eccentricity formula with correct substitution to find a value for b^2 or b. Could be implied. May see $b=a\sqrt{e^2-1}$ or use of e.g., $e=\sqrt{1+\frac{b^2}{a^2}}$ or $e=\frac{c}{a}$ with $c=\sqrt{a^2+b^2}$	M1
	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{36}-\frac{y^2}{\frac{25}{4}}=1$	Applies $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **correctly** for their values. Not dependent. Could use e.g., $b^2x^2-a^2y^2=a^2b^2$	M1
	e.g., $25x^2-144y^2=900$ A correct **equation** in correct form. Requires all previous 5 marks but allow if 4 marks with A0 in (a) for $e=\pm\frac{13}{12}$ and negative value not rejected in part (a). Any positive integer multiple. Allow equivalents provided variables on one side and constant on the other and y^2 term has negative coefficient. Just $p=25,\ q=144,\ r=900$ requires $px^2-qy^2=r$ to be seen. Ignore wrong values for p, q, r following a correct equation (e.g., "$q=-144$")		A1
Alt **Using** $PS^2=e^2PM^2$	$\left(x-\frac{13}{2}\right)^2+y^2=\left(\frac{13}{12}\right)^2\left(x-\frac{72}{13}\right)^2$ M1: Forms equation correct for their ae, e and $\frac{a}{e}$ $x^2-13x+\frac{169}{4}+y^2=\frac{169}{144}x^2-13x+36 \Rightarrow \frac{25}{144}x^2-y^2=\frac{25}{4}$ M1: Expands and reaches $rx^2-sy^2=t,\ \ r,s,t\neq 0$ A1: e.g., $25x^2-144y^2=900$ as main scheme		
			(3)
			Total 6

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td>2(a)</td><td>$\det(\mathbf{M}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda & 0 & 3\\ 0 & -4-\lambda & -3\\ 0 & -4 & -\lambda\end{vmatrix}$
$=$ e.g., $(2-\lambda)[(-4-\lambda)(-\lambda)-(-4)(-3)]-0+3(0)$
or $(2-\lambda)[(-4-\lambda)(-\lambda)-(-4)(-3)]-0+0$
Sarrus $\Rightarrow (2-\lambda)(-4-\lambda)(-\lambda)-(2-\lambda)(-3)(-4)$</td><td>Obtains an unsimplified cubic expression for $\det(\mathbf{M}-\lambda\mathbf{I})$ condoning sign/copying slips only. Allow poor bracketing if intention clear.</td><td>M1</td></tr>
<tr><td></td><td colspan="2">Note: It is possible to just use $\mathbf{M}\boldsymbol{x}=\lambda\boldsymbol{x}$ e.g.,
$-4y=\lambda z\Rightarrow y=-\frac{\lambda z}{4}$ and $-4y-3z=\lambda y\Rightarrow \lambda z-3z=-\frac{\lambda^2 z}{4}\Rightarrow \lambda^2+4\lambda-12=0\Rightarrow ...$
Score the M1 for achieving a 3TQ in λ from appropriate work condoning copying/sign slips only</td><td></td></tr>
<tr><td></td><td colspan="2">$(2-\lambda)(\lambda^2+4\lambda-12)=0$ or $\lambda^3+2\lambda^2-20\lambda+24=0$ or $-\lambda^3-2\lambda^2+20\lambda-24=0$
$(2-\lambda)(\lambda-2)(\lambda+6)=0$ or $(\lambda+6)(\lambda-2)(\lambda-2)=0$
$\lambda_1=-6\ \ (\lambda_2=2)$
dM1: Solves $\det(\mathbf{M}-\lambda\mathbf{I})=0$ to obtain any value for λ including 2. Not usual rules – award for any value seen that is consistent with their equation.
The "=0" can be implied by a solution.
Note that they may disregard the $(2-\lambda)$ and solve a quadratic.
A1: –6 from a correct equation. Accept both solutions e.g., "-6, 2" and allow if mislabelled and/or -6 rejected. No incorrect solutions.</td><td>dM1
A1</td></tr>
<tr><td></td><td>$\mathbf{M}\boldsymbol{x}=-6\boldsymbol{x}\ \Rightarrow\ \begin{array}{c}2x+3z=-6x\\ -4y-3z=-6y\\ -4y=-6z\end{array}\ \Rightarrow x=...,\ y=...,\ z=...$
$(\mathbf{M}+6\mathbf{I})\boldsymbol{x}=\mathbf{0}\ \Rightarrow\ \begin{array}{c}8x+3z=0\\ 2y-3z=0\\ -4y+6z=0\end{array}\ \Rightarrow x=...,\ y=...,\ z=...$</td><td>Uses $\mathbf{M}\boldsymbol{x}=\lambda\boldsymbol{x}$ or $(\mathbf{M}-\lambda\mathbf{I})\boldsymbol{x}=0$ with any of their non-zero eigenvalues (however obtained) to form simultaneous equations and solves. No requirement for a vector for this mark. There is no need to check their values but award M0 for a zero solution.</td><td>M1</td></tr>
<tr><td></td><td colspan="2">Note: Could find vector product of first 2 rows of $\mathbf{M}-\lambda\mathbf{I}$ i.e.,
$(8\mathbf{i}+3\mathbf{k})\times(2\mathbf{j}-3\mathbf{k})=(-6\mathbf{i}+24\mathbf{j}+16\mathbf{k})$ (two correct components)</td><td></td></tr>
<tr><td></td><td>$\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-3\\12\\8\end{pmatrix}\Rightarrow\frac{1}{\sqrt{3^2+12^2+8^2}}\begin{pmatrix}-3\\12\\8\end{pmatrix}$</td><td>Correct method to normalise their eigenvector no matter how this vector is obtained provided it has at least 2 non-zero components. Only allow slips if there is working.</td><td>M1</td></tr>
<tr><td></td><td>e.g., $\frac{1}{\sqrt{217}}\begin{pmatrix}-3\\12\\8\end{pmatrix}$ or $\begin{pmatrix}-\frac{3\sqrt{217}}{217}\\ \frac{12\sqrt{217}}{217}\\ \frac{8\sqrt{217}}{217}\end{pmatrix}$ or $\begin{pmatrix}-\frac{3}{\sqrt{217}}\\ \frac{12}{\sqrt{217}}\\ \frac{8}{\sqrt{217}}\end{pmatrix}$ or $\frac{1}{2\sqrt{217}}\begin{pmatrix}-6\\24\\16\end{pmatrix}$</td><td>A correct normalised eigenvector in any form. Note direction may be reversed.
May use $\mathbf{i},\ \mathbf{j},\ \mathbf{k}$ notation</td><td>A1</td></tr>
<tr><td></td><td colspan="2"></td><td>(6)</td></tr>
</table>

Question Number	Scheme	Notes	Marks
2(b)	May use **i**, **j**, **k** notation		
	Multiplies position and direction by **M** (not e.g., $\mathbf{M}-\lambda\mathbf{I}$) In parametric form: $\begin{pmatrix}2&0&3\\0&-4&-3\\0&-4&0\end{pmatrix}\begin{pmatrix}4+2\mu\\-1\\-\mu\end{pmatrix}=\ldots \quad \left\{\begin{pmatrix}8+4\mu-3\mu\\4+3\mu\\4\end{pmatrix}=\begin{pmatrix}8\\4\\4\end{pmatrix}+\mu\begin{pmatrix}1\\3\\0\end{pmatrix}\right\}$ There is no requirement to extract the vectors if parametric form is used. Allow this mark if e.g., $8+4\mu-3\mu$ written as $2(4+2\mu)-3\mu$ Allow this work without a parameter i.e., $\begin{pmatrix}2&0&3\\0&-4&-3\\0&-4&0\end{pmatrix}\begin{pmatrix}4\\-1\\0\end{pmatrix}=\ldots \left\{\begin{pmatrix}8\\4\\4\end{pmatrix}\right\}$ **and** $\begin{pmatrix}2&0&3\\0&-4&-3\\0&-4&0\end{pmatrix}\begin{pmatrix}2\\0\\-1\end{pmatrix}=\ldots \left\{\begin{pmatrix}1\\3\\0\end{pmatrix}\right\}$ **or** $\begin{pmatrix}2&0&3\\0&-4&-3\\0&-4&0\end{pmatrix}\begin{pmatrix}4&2\\-1&0\\0&-1\end{pmatrix}=\ldots \left\{\begin{pmatrix}8&1\\4&3\\4&0\end{pmatrix}\right\}$ Alternatively: Could find 2 points on l_1, transform them both and **subtract** to find direction. Allow slips and condone the matrix product written the wrong way round provided they have attempted to multiply the elements appropriately and they obtain a vector (or 3 x 2 matrix) with the resulting values correctly placed. Condone if they proceed to confuse which is the position and which is the direction.		**M1**
	$\mathbf{r}\times\begin{pmatrix}1\\3\\0\end{pmatrix}=\begin{pmatrix}8\\4\\4\end{pmatrix}\times\begin{pmatrix}1\\3\\0\end{pmatrix}$	Forms: $\mathbf{r}\times$ direction = position $\times$ direction Must not clearly confuse their vectors. Allow if RHS = direction x position. **Requires previous M mark.** No requirement to calculate vector product but the RHS could be implied by 2 correct components (or the negative version if the product is reversed)	**dM1**
	$\mathbf{r}\times\begin{pmatrix}1\\3\\0\end{pmatrix}=\begin{pmatrix}-12\\4\\20\end{pmatrix}$	Any correct equation in the correct form. Not $\mathbf{b}=\ldots$, $\mathbf{c}=\ldots$ unless $\mathbf{r}\times\mathbf{b}=\mathbf{c}$ seen. Isw once a correct answer is seen.	**A1**
			(3)
			Total 9

Question Number	Scheme	Notes	Marks
3(a)	$y=\text{arsinh}\left(\sqrt{x^2-1}\right)$		
	For all Ways allow the final answer to be written as $\dfrac{1}{\left(x^2-1\right)^{\frac{1}{2}}}$ or $\left(x^2-1\right)^{-\frac{1}{2}}$		
Way 1	$\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1+\left(\sqrt{x^2-1}\right)^2}}\times\dfrac{1}{2}\left(x^2-1\right)^{-\frac{1}{2}}(2x)$ M1: Obtains $\dfrac{1}{\sqrt{1+\left(\sqrt{x^2-1}\right)^2}}\times f(x)$ or e.g., $\dfrac{1}{x}\times f(x)$ $\quad f(x)\neq k$ A1: Fully correct unsimplified expression		**M1** **A1**
	$=\dfrac{1}{\sqrt{1+x^2-1}}\times\dfrac{x}{\sqrt{x^2-1}}=\dfrac{1}{\sqrt{x^2-1}}$ * or e.g., $=\dfrac{1}{x}\times\dfrac{x}{\sqrt{x^2-1}}=\dfrac{1}{\sqrt{x^2-1}}$ *	Correct completion with intermediate line of working and no errors	**A1***
			(3)
Way 2 **Takes sinh of both sides**	$y=\text{arsinh}\left(\sqrt{x^2-1}\right)\Rightarrow\sinh y=\sqrt{x^2-1}\Rightarrow\cosh y\dfrac{dy}{dx}=\dfrac{1}{2}\left(x^2-1\right)^{-\frac{1}{2}}(2x)$ M1: Takes sinh of both sides and differentiates to obtain $\cosh y\dfrac{dy}{dx}=f(x)$ $\quad f(x)\neq k$ A1: Fully correct unsimplified equation		**M1** **A1**
	$\cosh y=\sqrt{1+\sinh^2 y}$ or $\sqrt{1+x^2-1}\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{\sqrt{x^2-1}}$ *	Correct completion with clear use of identity (must see more than just $\cosh y=x$) and no errors	**A1***
			(3)
Way 3 **Takes sinh & squares**	$y=\text{arsinh}\left(\sqrt{x^2-1}\right)\Rightarrow\sinh y=\sqrt{x^2-1}\Rightarrow\sinh^2 y=x^2-1\Rightarrow 2\sinh y\cosh y\dfrac{dy}{dx}=2x$ M1: Takes sinh of both sides, squares and differentiates to obtain $c\sinh y\cosh y\dfrac{dy}{dx}=f(x)$ $\quad f(x)\neq k$ A1: Fully correct unsimplified expression or equation		**M1** **A1**
	$\cosh y=\sqrt{1+\sinh^2 y}$ or $\sqrt{1+x^2-1}\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{\sqrt{x^2-1}}$	Correct completion with clear use of identity (must see more than just $\cosh y=x$) and no errors	**A1***
			(3)
Way 4 **Takes sinh & squares & uses identity**	$\Rightarrow\sinh y=\sqrt{x^2-1}\Rightarrow\sinh^2 y=x^2-1\Rightarrow\cosh^2 y=1+\left(x^2-1\right)\Rightarrow\cosh^2 y=x^2\Rightarrow 2\sinh y\cosh y\dfrac{dy}{dx}=2x$ M1: Takes sinh of both sides, squares, uses identity and differentiates to obtain $c\sinh y\cosh y\dfrac{dy}{dx}=f(x)$ $\quad f(x)\neq k$ Allow sign errors with identity for the M mark. A1: Fully correct unsimplified expression or equation		**M1** **A1**
	$\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{\sqrt{x^2-1}}$	Correct completion with clear use of identity and no errors	**A1***
			(3)

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td>3(a)
Way 5

Takes sinh & squares & uses identity & roots</td><td colspan="2">$\Rightarrow \sinh y = \sqrt{x^2-1} \Rightarrow \sinh^2 y = x^2-1 \Rightarrow \cosh^2 y = 1+\left(x^2-1\right) \Rightarrow \cosh y = x \Rightarrow \sinh y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$
M1: Takes sinh of both sides, squares, uses identity, roots and differentiates to obtain $c\sinh y\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)$ or k
Allow sign errors with identity.
A1: Fully correct unsimplified expression or equation</td><td>M1
A1</td></tr>
<tr><td></td><td>$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2-1}}$</td><td>Correct completion with clear use of identity and no errors</td><td>A1*</td></tr>
<tr><td colspan="4" align="right">(3)</td></tr>
<tr><td>Way 6

Uses log form of arsinh first</td><td colspan="2">$y = \text{arsinh}\left(\sqrt{x^2-1}\right) \Rightarrow y = \ln\left(\sqrt{x^2-1}+\sqrt{x^2-1+1}\right) = \ln\left(\sqrt{x^2-1}+x\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{1}{2}\left(x^2-1\right)^{-\frac{1}{2}}(2x)+1}{\sqrt{x^2-1}+x}$
M1: Use log form of arsinh correctly and differentiates to obtain $\frac{\mathrm{f}(x) \neq k}{\sqrt{x^2-1}+x}$
A1: Fully correct unsimplified expression</td><td>M1
A1</td></tr>
<tr><td></td><td>$= \frac{\frac{x}{\sqrt{x^2-1}}+1}{\sqrt{x^2-1}+x}$ or $\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}} \times \frac{1}{\sqrt{x^2-1}+x} = \frac{1}{\sqrt{x^2-1}}$ *</td><td>Correct completion with intermediate line of working and no errors</td><td>A1*</td></tr>
<tr><td colspan="4" align="right">(3)</td></tr>
<tr><td colspan="4">You may see other variations e.g., using exponential definitions, attempts via dx/dy. The M mark is for differentiating to obtain correct forms and the first A is awarded if it is correct. The final A is for correct completion.</td></tr>
</table>

Question Number	Scheme	Notes	Marks
3(b)	$\mathrm{f}(x)=\frac{1}{3}\operatorname{arsinh}\left(\sqrt{x^2-1}\right)-\arctan x$		
	$\mathrm{f}'(x)=\frac{1}{3\sqrt{x^2-1}}-\frac{1}{1+x^2}$	$\mathrm{f}'(x)=\frac{A}{\sqrt{x^2-1}}\pm\frac{1}{1\pm x^2}$ $A=\frac{1}{3}$, 3 or 1	**M1 (B1 on ePen)**
	$1+x^2=3\sqrt{x^2-1}$ $1+2x^2+x^4=9x^2-9$	Sets $\frac{A}{\sqrt{x^2-1}}\pm\frac{1}{1+x^2}=0$ $A=\frac{1}{3}$, 3 or 1 **Denominator of derivative of arctan x must now be $1+x^2$** Cross multiplies and squares to obtain the correct form for both sides so do not condone e.g., $\left(1+x^2\right)^2=1+x^4$ May see the quartic obtained through equivalent work.	**M1**
	$x^4-7x^2+10=0\Rightarrow\left(x^2-2\right)\left(x^2-5\right)=0\Rightarrow x^2=2,\ 5$ Solves a 3TQ in x^2 (usual rules and one correct root if no working). No requirement to see the terms collected. Ignore labelling of solutions so allow e.g., "$\underline{x}=2,\ 5$". One correct value for their equation if no working, which may be for x or x^2, so just look for the values. May change the variable. Allow for a correct solution with no working from solving a three term quartic of the correct form on a calculator. Allow if value for x^2 is negative or if roots are complex. **Requires previous M marks.**		**ddM1**
	$x=\sqrt{2},\ \sqrt{5}$	Both exact and no other solutions e.g., ± is A0 unless negatives rejected. Must not reject either correct solution.	**A1**
			(4)
			Total 7

Question Number	Scheme/Notes	Marks
4(a)	$\sinh(A+B) = \sinh A\cosh B + \cosh A\sinh B$	
	There is no credit for proofs that do not use exponential definitions	
	$\{\sinh A\cosh B + \cosh A\sinh B =\}$ $\dfrac{e^A - e^{-A}}{2}\times\dfrac{e^B + e^{-B}}{2} + \dfrac{e^A + e^{-A}}{2}\times\dfrac{e^B - e^{-B}}{2}$ or e.g., $\dfrac{(e^A - e^{-A})(e^B + e^{-B}) + (e^A + e^{-A})(e^B - e^{-B})}{4}$ Replaces two of the four hyperbolic functions with correct exponential expressions. Condone poor bracketing. If they immediately start expanding this mark is only implied by completely correct work (i.e., with exponential definitions correct) and not just the fractions shown in the A1* note	**M1**
	$= \dfrac{e^{A+B} - e^{B-A} + e^{A-B} - e^{-A-B} + e^{A+B} + e^{B-A} - e^{A-B} - e^{-A-B}}{4}$ Expands numerator (or numerators if 2 separate fractions). Allow for sign errors only with coefficients and indices **and must see at least four terms** (but count terms which have been crossed out by cancelling) Allow this mark for: $= \dfrac{e^A e^B - e^{-A}e^B + e^A e^{-B} - e^{-A}e^{-B} + e^A e^B + e^{-A}e^B - e^A e^{-B} - e^{-A}e^{-B}}{4}$ Must see at least four terms as before but the last mark will not be available unless the requirements shown below are satisfied.	**M1**
	$= \dfrac{2e^{A+B} - 2e^{-(A+B)}}{4}$ or $\dfrac{2(e^{A+B} - e^{-(A+B)})}{4}$ or $\dfrac{e^{A+B} - e^{-(A+B)}}{2}$ or $\dfrac{1}{2}(e^{A+B} - e^{-(A+B)})$ or $\dfrac{e^{A+B}}{2} - \dfrac{e^{-(A+B)}}{2}$ $= \sinh(A+B)$ * Reaches $\sinh(A+B)$ with no errors. Condone if the "$\sinh A\cosh B + \cosh A\sinh B =$" is missing at the start but the "$= \sinh(A+B)$" or "= LHS" must be seen. All bracketing correct where required but condone an unclosed bracket. One of the expressions shown or similar must be seen and allow –A–B used for $-(A+B)$. Allow a "meet in the middle" proof and condone a "1=1" style approach provided it is complete. In both these cases a minimal conclusion is required e.g., "shown" but allow if both "LHS = ..." and "...=RHS" are seen. Do not condone sinh and/or cosh written as sin/cos for this mark	**A1***
	Attempts that start with the LHS and do not revert to a "meet in the middle" approach: Score the second M provided an **eight** term expanded numerator is achieved. The first M is for two explicitly clear correct replacements of hyperbolic expressions with two of $\sinh A, \cosh B, \cosh A$ and $\sinh B$. Condone if the $\sinh(A+B) =$ is missing at the start in these cases but the RHS or "...=RHS" must be seen.	
		(3)

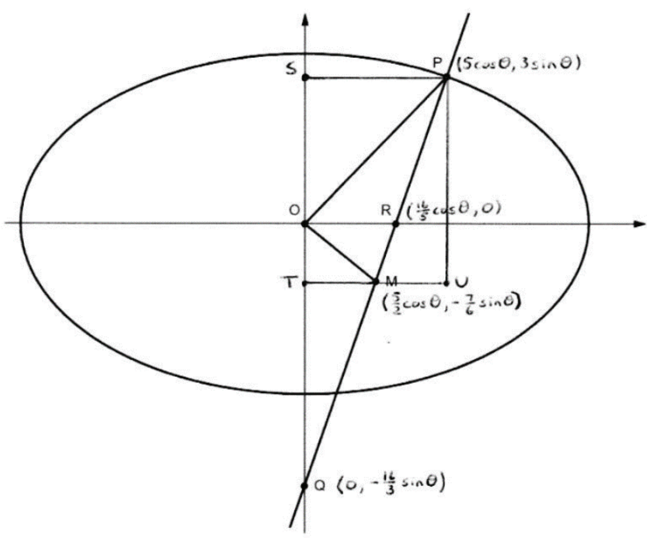
Question Number	Scheme	Notes	Marks
4(b)	Condone the use of e.g., B for α or k for R for the first three marks but allow the A mark if recovered which may be via a correct expression which might be in (c)		
	$10\sinh x+8\cosh x=R\sinh x\cosh\alpha+R\cosh x\sinh\alpha$ $\Rightarrow R\sinh\alpha=8,\quad R\cosh\alpha=10$ Equates coefficients to obtain the two correct equations. This mark could be implied by either **correct** elimination, i.e., $R^2=10^2-8^2$ or $\tanh\alpha=\frac{8}{10}$ **provided incorrect equations are not seen.**		**B1 (M1 on ePen)**
	A complete attempt at finding a positive value for R: **By elimination**: $R^2\left(\cosh^2\alpha-\sinh^2\alpha\right)=10^2-8^2\Rightarrow R^2=36\Rightarrow R=6$ Allow this mark for $R=\sqrt{10^2+8^2}=2\sqrt{41}$ or $\sqrt{164}$. May just see e.g., $R=2\sqrt{41}$ **Following a positive value obtained for α where $\alpha=k\ln p,\ k>0,\ p>1$:** $\alpha=\frac{1}{2}\ln 9=\ln 3\Rightarrow R\cosh(\ln 3)=10\Rightarrow R\left(\frac{e^{\ln 3}+e^{-\ln 3}}{2}\right)=10\Rightarrow R=\ldots\quad\left\{\frac{5}{3}R=10\Rightarrow R=6\right\}$ or $R\sinh(\ln 3)=8\Rightarrow R\left(\frac{e^{\ln 3}-e^{-\ln 3}}{2}\right)=8\Rightarrow R=\ldots\quad\left\{\frac{4}{3}R=8\Rightarrow R=6\right\}$ Correct exponential definitions must be used but can be implied by correct work. Allow if the 10 and 8 are mixed up and allow slips in solving		**1st M1**
	A complete attempt at finding a positive value for α where $\alpha=k\ln p,\ k>0,\ p>1$: **By elimination:** $\tanh\alpha=\frac{8}{10}\Rightarrow\alpha=\text{artanh}\left(\frac{4}{5}\right)=\frac{1}{2}\ln\left(\frac{1+\frac{4}{5}}{1-\frac{4}{5}}\right)=\ldots\left\{=\frac{1}{2}\ln 9=\ln 3\right\}$ A correct logarithmic form must be used with a valid value for artanh (<1) **Following a positive value obtained for R:** $\sinh\alpha=\frac{8}{\text{"6"}}\Rightarrow\alpha=\text{arsinh}\left(\frac{8}{\text{"6"}}\right)=\ln\left(\frac{8}{\text{"6"}}+\sqrt{\left(\frac{8}{\text{"6"}}\right)^2+1}\right)\quad\{=\ln 3\}$ $\cosh\alpha=\frac{10}{\text{"6"}}\Rightarrow\alpha=\text{arcosh}\left(\frac{10}{\text{"6"}}\right)=\ln\left(\frac{10}{\text{"6"}}+\sqrt{\left(\frac{10}{\text{"6"}}\right)^2-1}\right)\quad\{=\ln 3\}$ A correct logarithmic form must be used with a valid value if using arcosh (>1) The appropriate logarithmic forms could be implied by correct values. Allow this mark if e.g., $\frac{8}{10}$ is erroneously simplified but the value must be valid for the inverse hyperbolic function. If an exponential form is used to evaluate an inverse hyperbolic the form must be correct and the solving of any resulting 3TQ (most likely in e^{α} or e^{x}) must satisfy usual rules with one root correct if no working. Note that using tanh leads to a 2TQ which they must get one correct root for. They must also proceed to $\alpha=k\ln p,\ k>0,\ p>1$		**2nd M1**
	$6\sinh(x+\ln 3)$ or $R=6$ and $\alpha=\ln 3$ (or $p=3$) Correct expression but allow values for R and α (or p). If all the values are not seen in (b) then allow if they are seen in (c) and they could be seen embedded in a correct expression. A0 for additional solutions e.g., $6\sinh(x\pm\ln 3)$		**A1**

Question Number	Scheme/Notes	Marks
4(c)	There is no credit for attempts that do not use part (b) so e.g., do not award marks for attempts that apply exponential definitions to $10\sinh x + 8\cosh x = 18\sqrt{7}$ but note that it is acceptable to use exponential definitions with $6\sinh(x+\ln 3) = 18\sqrt{7}$. Allow work with "made up" values for R and p provided $R > 0$, $p \in \mathbb{Z}$, $p > 1$	
	$6\sinh(x+\ln 3) = 18\sqrt{7}$ $\Rightarrow x = \text{arsinh}(3\sqrt{7}) - \ln 3$ $\Rightarrow x = \ln\left(3\sqrt{7} + \sqrt{(3\sqrt{7})^2 + 1}\right) - \ln 3$ Obtains $x = \text{arsinh}\left(\frac{18\sqrt{7}}{"6"}\right) \pm \ln"3"$ or $x \pm \ln"3" = \text{arsinh}\left(\frac{18\sqrt{7}}{"6"}\right)$ from $"6"\sinh(x \pm \ln"3") = 18\sqrt{7}$ **and** uses the correct logarithmic form to obtain an expression for, or equation in x in "ln"s only but condone loss of the –ln "3" or +ln"3" after it has been seen. If the –ln "3" or +ln"3" is immediately incorporated to make a single logarithm the subtraction/addition law must be applied correctly. Work must be exact and not in decimals. If e.g., $C = \text{arsinh}(3\sqrt{7})$ is found using $\frac{e^C - e^{-C}}{2} = 3\sqrt{7}$, the exponential definition must be correct and they must solve a 3TQ in e^C satisfying usual rules (or one root correct if no working) and proceed to a valid $C = ...$ (e.g., not ln(negative)). This also applies to attempts via $6\frac{e^{x+\ln 3} - e^{-x-\ln 3}}{2} = 18\sqrt{7}$ $\left\{\Rightarrow 3e^x - \frac{1}{3}e^{-x} = 6\sqrt{7} \Rightarrow 9e^{2x} - 18\sqrt{7}e^x - 1 = 0 \Rightarrow x = \ln\left(\frac{8+3\sqrt{7}}{3}\right)\right\}$ Note that $e^{2(x+\ln 3)} - 6\sqrt{7}e^{x+\ln 3} - 1 = 0 \Rightarrow e^{x+\ln 3} = 8 + 3\sqrt{7} \Rightarrow x = \ln\left(\frac{8+3\sqrt{7}}{3}\right)$ is also possible and in such cases the x + ln "3" must be handled correctly	M1
	$\left\{x = \ln\left(\frac{3\sqrt{7}+8}{3}\right) = \right\}\ln\left(\sqrt{7} + \frac{8}{3}\right)$ Correct answer in correct form. Accept e.g., $\ln\left(2\frac{2}{3} + \sqrt{7}\right)$. Must be fully bracketed correctly. Accept $q = \frac{8}{3}$ if $\ln\left(\sqrt{7} + q\right)$ is seen. No additional answers.	A1
		(2)
		Total 9

Question Number	Scheme	Notes	Marks
5(a)	$4x^2+4x+17=4\left(x^2+x+\frac{17}{4}\right)=4\left[\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+\frac{17}{4}\right]=(2x+1)^2+16$ or $4x^2+4x+17=4x^2+4px+q\Rightarrow 4px=4x\Rightarrow \underline{p=1},\ q+p^2=17\Rightarrow \underline{q=16}$ B1: Either p or q correct B1: Both correct values in part (a). Allow from any/no work. Values may be embedded within expression $(2x+p)^2+q$.		**B1** **B1**
			(2)
(b)	$A=8,\ B=4$	Both correct values (accept if embedded)	**B1**
			(1)
(c)	Note that this is a Hence question and there is no credit for work on the original fraction		
	$\int\frac{8x+5}{\sqrt{4x^2+4x+17}}\mathrm{d}x=\int\frac{1}{\sqrt{(2x+1)^2+16}}\mathrm{d}x+\int\frac{8x+4}{\sqrt{4x^2+4x+17}}\mathrm{d}x$ $=\frac{1}{2}\operatorname{arsinh}\left(\frac{2x+1}{4}\right)+2\left(4x^2+4x+17\right)^{\frac{1}{2}}$ or $\frac{1}{2}\ln\left(\frac{2x+1}{4}+\sqrt{\left(\frac{2x+1}{4}\right)^2+1}\right)+2\left(4x^2+4x+17\right)^{\frac{1}{2}}$ or $\frac{1}{2}\ln\left(2x+1+\sqrt{(2x+1)^2+16}\right)+2\left((2x+1)^2+16\right)^{\frac{1}{2}}$	M1: For ...$\operatorname{arsinh}(\mathrm{f}(x))$ $\mathrm{f}(x)\neq k$ or logarithmic equivalent i.e., $...\ln\left(\mathrm{f}(x)+\sqrt{(\mathrm{f}(x))^2+c}\right)$ $c\neq 0$ M1: For $...\left(4x^2+4x+17\right)^{\frac{1}{2}}$ or $...\left((2x+1)^2+16\right)^{\frac{1}{2}}$ A1: Fully correct integration	**M1** **M1** **A1**
	Allow for equivalents in e.g., u if substitutions are used e.g., $u=2x+1\Rightarrow\frac{1}{2}\int\frac{1}{\sqrt{u^2+16}}\mathrm{d}u\Rightarrow\frac{1}{2}\operatorname{arsinh}\left(\frac{u}{4}\right)$ $u=4x^2+4x+17\Rightarrow\int\frac{1}{\sqrt{u}}\mathrm{d}u\Rightarrow 2\sqrt{u}$ $4\sinh u=2x+1\Rightarrow\int\frac{2\cosh u}{\sqrt{16\cosh^2 u}}\mathrm{d}u\Rightarrow\frac{1}{2}\int\mathrm{d}u=\frac{1}{2}u$ Score the M marks for appropriate forms (sign/coefficient errors only). If they continue working in terms of u the limits applied for the **dd**M1 must be correct for their substitution which for the above examples would be 3 & $\frac{5}{3}$, 25 & $\frac{169}{9}$ and $\operatorname{arsinh}\left(\frac{3}{4}\right)$ & $\operatorname{arsinh}\left(\frac{5}{12}\right)$		
	$\int_{\frac{1}{3}}^{1}\frac{8x+5}{\sqrt{4x^2+4x+17}}\mathrm{d}x=\frac{1}{2}\operatorname{arsinh}\left(\frac{3}{4}\right)-\frac{1}{2}\operatorname{arsinh}\left(\frac{5}{12}\right)+2\sqrt{25}-2\sqrt{\frac{169}{9}}$ $\Rightarrow\frac{1}{2}\ln\left(\frac{3}{4}+\sqrt{\left(\frac{3}{4}\right)^2+1}\right)-\frac{1}{2}\ln\left(\frac{5}{12}+\sqrt{\left(\frac{5}{12}\right)^2+1}\right)+2\sqrt{25}-\frac{26}{3}$ Condone replacement of $\operatorname{arsinh}\left(\frac{x}{a}\right)$ with $\ln\left(x+\sqrt{x^2+a^2}\right)$ with $a\neq 1$ instead of using $\operatorname{arsinh}x=\ln\left(x+\sqrt{x^2+1}\right)$	Substitutes and subtracts with the given limits and uses the appropriate form for arsinh twice (if required). Results from separate integrals must be combined. Allow slips but the $\mathrm{f}\left(\frac{1}{3}\right)$ terms (and no others) must be subtracted. Not implied by just the final answer. **Requires both previous M marks.**	**ddM1**
	arsinh(...) may be evaluated using correct exp definition & solving a exponential 3TQ		
	$\left\{=\frac{1}{2}\ln 2-\frac{1}{2}\ln\frac{3}{2}+10-\frac{26}{3}\right\}=\frac{4}{3}+\frac{1}{2}\ln\frac{4}{3}$ Correct answer in correct form. May be no further work following substitution but there must be nothing incorrect. Allow $k=\frac{4}{3}$ if $k+\frac{1}{2}\ln k$ is seen. Allow $\frac{1}{2}\ln\frac{4}{3}+\frac{4}{3}$		**A1**
	Algebraic integration must be used. Answer or 1.47717... only scores no marks		
			(5)
			Total 8

Question Number	Scheme	Notes	Marks
6(a)	$\frac{x^2}{25}+\frac{y^2}{9}=1 \quad P(5\cos\theta, 3\sin\theta)$		
	$\left\{\frac{dx}{d\theta}=-5\sin\theta \quad \frac{dy}{d\theta}=3\cos\theta\right\}$ $\frac{dy}{dx}=-\frac{3\cos\theta}{5\sin\theta}$; $\frac{2x}{25}+\frac{2y}{9}\frac{dy}{dx}=0$ $\frac{dy}{dx}=-\frac{9x}{25y}\left\{=-\frac{45\cos\theta}{75\sin\theta}\right\}$	$y=\left(9-\frac{9}{25}x^2\right)^{\frac{1}{2}}$ $\frac{dy}{dx}=\frac{-\frac{18}{25}x}{2\sqrt{9-\frac{9}{25}x^2}}\left\{=\frac{-\frac{18}{25}\times 5\cos\theta}{2\sqrt{9-9\cos^2\theta}}\right\}$	B1
	Any correct expression for $\frac{dy}{dx}$ in terms of θ, or x and y, or x. Allow for a correct $\frac{dx}{dy}$ or $-\frac{dx}{dy}$		
	$m_T=-\frac{3\cos\theta}{5\sin\theta}\Rightarrow m_N=\frac{5\sin\theta}{3\cos\theta}$	Correct perpendicular gradient rule for their $\frac{dy}{dx}$ **in terms of** θ May see $m_T=-\frac{3}{5}\cot\theta\Rightarrow m_N=\frac{5}{3}\tan\theta$	M1
	$y-3\sin\theta=\frac{5\sin\theta}{3\cos\theta}(x-5\cos\theta)$ OR $y=mx+c\Rightarrow 3\sin\theta=\frac{5\sin\theta}{3\cos\theta}\times 5\cos\theta+c\Rightarrow c=-\frac{16}{3}\sin\theta$ Correct straight line method with a changed gradient in terms of θ		M1
	$3y\cos\theta-9\sin\theta\cos\theta=5x\sin\theta-25\sin\theta\cos\theta$ $\Rightarrow 5x\sin\theta-3y\cos\theta=16\sin\theta\cos\theta$ *	Reaches given answer with intermediate line of working and no errors. Allow this equation written in reverse, x and y terms in different order provided they are together with the third term on the other side and allow the products in a different order provided the numerical coefficients "5", "–3" and "16" are at the front of the terms.	A1*
	The last three marks require $P(5\cos\theta, 3\sin\theta)$ to be substituted but condone using e.g, $\frac{25y}{9x}$ as the normal gradient when forming the straight line <u>provided</u> appropriate substitution is seen before the given answer.		
			(4)

Question Number	Scheme	Notes	Marks
6(b)	At Q, $x=0 \Rightarrow y=-\frac{16}{3}\sin\theta$	**Correct** y coordinate of Q. Accept unsimplified	**B1**
	M is $\left(\frac{5\cos\theta+0}{2}, \frac{3\sin\theta+'-\frac{16}{3}\sin\theta'}{2}\right)$ Accept $x=\frac{5}{2}\cos\theta,\ y=-\frac{7}{6}\sin\theta$	Correct method for midpoint for both coordinates with their y_Q. Could be implied. **Alternatively**, award for $\Delta OPM=\frac{1}{2}\Delta OPQ=\frac{1}{2}\times\frac{1}{2}\times\frac{16}{3}\sin\theta\times5\cos\theta$ (see area examples below)	**M1**
	e.g., PQ meets x-axis at $R\left(\frac{16}{5}\cos\theta,\ 0\right)$ $\Rightarrow$ Area $\Delta OPM=\Delta OPR+\Delta OMR$ $=\frac{1}{2}\times\frac{16}{5}\cos\theta\left(3\sin\theta+\frac{7}{6}\sin\theta\right)$	Correct unsimplified expression for area of ΔOPM Do not allow recovery from a negative area. **Can only follow incorrect work** i.e., **an incorrect midpoint if** $\Delta OPM=\frac{1}{2}\Delta OPQ$ **is used.** Please see below for alternatives	**M1**
	If shoelace method is used, score for a correct "extracted" expression for the area (allow with modulus if correct) e.g., $\frac{1}{2}\begin{vmatrix}0 & 5\cos\theta & \frac{5}{2}\cos\theta & 0\\ 0 & 3\sin\theta & -\frac{7}{6}\sin\theta & 0\end{vmatrix}$ $\Rightarrow\frac{1}{2}\left\|(5\cos\theta)\left(-\frac{7}{6}\sin\theta\right)-\left(\frac{5}{2}\cos\theta\right)(3\sin\theta)\right\|$ or $\frac{1}{2}\left[(5\cos\theta)\left(\frac{7}{6}\sin\theta\right)+\left(\frac{5}{2}\cos\theta\right)(3\sin\theta)\right]$		
	$\left\{=\frac{20}{3}\sin\theta\cos\theta=\frac{10}{3}\sin2\theta\right\}\Rightarrow(\text{area}=)\frac{10}{3}$	Correct area <u>following a correct expression</u>	**A1**
	$\frac{10}{3}$ <u>and justification</u>: **From** $\frac{10}{3}\sin2\theta$: max (value) of $\sin2\theta$ is 1 or e.g., $-1\leqslant\sin2\theta\leqslant1$ or states $\theta=\frac{\pi}{4}$ or 45° or obtains this using differentiation: $\left\{\frac{10}{3}\right\}\sin2\theta\Rightarrow\left\{\frac{20}{3}\right\}\cos2\theta=0\Rightarrow\ldots$ Do not accept if there is any wrong statement e.g., $\sin2\theta\leqslant1$, $-1<\sin2\theta<1$ but we will condone the ambiguous "$\sin2\theta$ is between 1 and -1" **From any other expression**: Must <u>differentiate</u> (unless rewrites as $\frac{10}{3}\sin2\theta$) e.g., $\frac{20}{3}\sin\theta\cos\theta$ $\frac{20}{3}\left(\cos^2\theta-\sin^2\theta\right)\Rightarrow\frac{20}{3}\cos2\theta=0$ or $\tan^2\theta=1\Rightarrow\theta=\frac{\pi}{4}$ or 45° Ignore any further differentiation to justify maximum		**A1**
			(5)
			Total 9



May see:

$\Delta OPM=\frac{1}{2}\Delta OPQ=\frac{1}{2}\times\frac{1}{2}\times\frac{16}{3}\sin\theta\times5\cos\theta$

(Scores the first 2 M marks together since M is not required – ignore an absent or wrong M)

$\Delta OPM=\Delta OPQ-\Delta OMQ$

$\frac{1}{2}\times\frac{16}{3}\sin\theta\times5\cos\theta-\frac{1}{2}\times\frac{16}{3}\sin\theta\times\frac{5}{2}\cos\theta$

$\Delta OPM=\Delta PQS-\Delta OMQ-\Delta PSO$

$=\frac{1}{2}\times\left(\frac{16}{3}\sin\theta+3\sin\theta\right)\times5\cos\theta-\frac{1}{2}\times\frac{16}{3}\sin\theta\times\frac{5}{2}\cos\theta-\frac{1}{2}\times3\sin\theta\times5\cos\theta$

$\left\{=\frac{125}{6}\sin\theta\cos\theta-\frac{20}{3}\sin\theta\cos\theta-\frac{15}{2}\sin\theta\cos\theta\right\}$

$\Delta OPM=PSTU-\Delta PSO-\Delta OMT-\Delta PMU$

$=5\cos\theta\times\left(3\sin\theta+\frac{7}{6}\sin\theta\right)-\frac{1}{2}\times3\sin\theta\times5\cos\theta$

$-\frac{1}{2}\times\frac{5}{2}\cos\theta\times\frac{7}{6}\sin\theta-\frac{1}{2}\times\left(5\cos\theta-\frac{5}{2}\cos\theta\right)\left(3\sin\theta+\frac{7}{6}\sin\theta\right)$

$\left\{=\left(\frac{125}{6}-\frac{15}{2}-\frac{35}{24}-\frac{125}{24}\right)\sin\theta\cos\theta\right\}$

Note that attempts that start by using Pythagoras for PM will also require the perpendicular distance from O to the line

Question Number	Scheme	Notes	Marks
7	$y=\ln\left(\tanh\frac{x}{2}\right) \qquad 1\leqslant x\leqslant 2$		
(a)	$\frac{dy}{dx}=\frac{1}{\tanh\frac{x}{2}}\times\frac{1}{2}\text{sech}^2\frac{x}{2}$ or e.g., $\frac{1}{2}\coth\frac{x}{2}\left(1-\tanh^2\frac{x}{2}\right)$ or $e^y=\tanh\frac{x}{2}\Rightarrow\left(\tanh\frac{x}{2}\right)\frac{dy}{dx}=\frac{1}{2}\text{sech}^2\frac{x}{2}$ or $\Rightarrow \text{artanh}\left(e^y\right)=\frac{x}{2}\Rightarrow\left(\frac{e^y}{1-e^{2y}}\right)\frac{dy}{dx}=\frac{1}{2}\Rightarrow\frac{dy}{dx}=\frac{1}{2}\coth\frac{x}{2}\left(1-\tanh^2\left(\frac{x}{2}\right)\right)$ Obtains an expression for (or equation involving) $\frac{dy}{dx}$ of appropriate form. Condone sign/coefficient errors only and any $\frac{x}{2}$s written as x but no "y"s. Do not condone missing "h"s in hyperbolic functions unless they are recovered		M1
	$\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx\Rightarrow\int\sqrt{1+\left(\frac{\text{sech}^2\frac{x}{2}}{2\tanh\frac{x}{2}}\right)^2}(dx)$ or e.g., $\sqrt{1+\left(\frac{dy}{dx}\right)^2}\Rightarrow\sqrt{1+\left(\frac{\cosh\frac{x}{2}}{2\sinh\frac{x}{2}\cosh^2\frac{x}{2}}\right)^2}$ Applies arc length formula (with or without the integration sign) with their $\frac{dy}{dx}$ which may have been simplified incorrectly before substitution. Do not condone attempts that clearly have worked backwards to deduce that the derivative is cosech x. Also condone incorrect work processing $1+\left(\frac{dy}{dx}\right)^2$ provided the expression is shown as square rooted afterwards. Not dependent. Ignore any multiplier such as π or 2π or k but forming $y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ is M0		M1
	$\sqrt{1+\left(\frac{1}{2\sinh\frac{x}{2}\cosh\frac{x}{2}}\right)^2}\rightarrow\sqrt{1+\left(\frac{1}{\sinh x}\right)^2}$ Uses identity/identities (sign errors only) to obtain $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ in terms of x and not $\frac{x}{2}$. Attempts that square the derivative and add the 1 first before attempting to convert to x must be convincing. **Requires both previous M marks.**		ddM1
	$\sqrt{1+\left(\frac{1}{\sinh x}\right)^2}=\sqrt{1+\text{cosech}^2 x}\Rightarrow s=\int_1^2\coth x\,dx$ or e.g., $=\int\sqrt{\frac{\sinh^2 x+1}{\sinh^2 x}}dx\Rightarrow s=\int_1^2\coth x\,dx$ Obtains given answer with no errors and at least one non-trivial intermediate line following the first expression in terms of x. Allow without "$s=$" but RHS must be exactly as printed. Allow a "meet in the middle" approach but it must be convincing and $\int_1^2\coth x\,dx$ must be seen. No missing "h"s in hyperbolic functions or missing arguments even if recovered. We will condone the occasional notational error e.g., $\text{sech}\frac{x}{2}^2$ if recovered		A1*
			(4)

Question Number	Scheme	Notes	Marks
7(b)	$\int \coth x \, dx = \ln(\sinh x)$ Correct integration. May see $-\ln(\operatorname{cosech} x)$ May see the sinh x in exponentials without the "2" which may come from the substitution $u = e^x - e^{-x}$ i.e., $\ln(e^x - e^{-x})$		B1
	1, 2 & 3. $\ln\left(\frac{e^2 - e^{-2}}{2}\right) - \ln\left(\frac{e - e^{-1}}{2}\right) = \ln\left(\frac{e^2 - \frac{1}{e^2}}{e - \frac{1}{e}}\right)$ or **4.** $\ln(\sinh 2) - \ln(\sinh 1) = \ln\left(\frac{\sinh 2}{\sinh 1}\right)$ Following replacement of $\int \coth x \, dx$ with $\pm\ln(\sinh x)$, $\pm\ln(\cosh x)$, $\pm\ln(\operatorname{cosech} x)$ or $\pm\ln(\operatorname{sech} x)$, substitutes given limits, subtracts and writes as a single logarithm. Condone sign errors if exponential forms used and may use negative powers of e.		M1
	1. $\ln\left(\frac{e^2 - \frac{1}{e^2}}{e - \frac{1}{e}}\right) = \ln\left(\frac{\frac{e^4 - 1}{e^2}}{\frac{e^2 - 1}{e}}\right) = \ln\left(\frac{e^4 - 1}{e^3 - e}\right)$ or **2.** $\Rightarrow \ln\left(\frac{\left(e + \frac{1}{e}\right)\left(e - \frac{1}{e}\right)}{e - \frac{1}{e}}\right)$ or $\ln\left(\frac{(e + e^{-1})(e - e^{-1})}{(e - e^{-1})}\right)$ or **3.** $\Rightarrow \ln\left(\frac{e^2 - e^{-2}}{e - e^{-1}} \times \frac{e + e^{-1}}{e + e^{-1}}\right) = \ln\left(\frac{(e^2 - e^{-2})(e + e^{-1})}{e^2 - e^{-2}}\right)$ or **4.** $\ln\left(\frac{\sinh 2}{\sinh 1}\right) = \ln\left(\frac{2\sinh 1 \cosh 1}{\sinh 1}\right)$ Following use of correct exponential form for sinh/cosech: **1.** Obtains a **correct** ln of a **single** fraction (or product of **single** fractions) with no negative powers of e **or** **2.** Uses difference of two squares to correctly factorise numerator **or** **3.** Applies correct multiplier to achieve expression shown **or** **4.** Correctly replaces $\sinh 2$ with $2\sinh 1 \cosh 1$ allowing equivalent work e.g., $\frac{\sinh 2}{\sinh 1} = \sqrt{\frac{(2\cosh^2 1 - 1)^2 - 1}{\cosh^2 1 - 1}} = \sqrt{\frac{4\cosh^4 1 - 4\cosh^2 1}{\cosh^2 1 - 1}} \Rightarrow s = \ln\sqrt{4\cosh^2 1}$ **Requires previous M mark.**		dM1
	1. $s = \ln\left(\frac{(e^2 + 1)(e^2 - 1)}{e(e^2 - 1)}\right) = \ln\left(e + \frac{1}{e}\right)$ or **2 & 3.** $s = \ln\left(e + \frac{1}{e}\right)$ or **4.** $s = \ln(2\cosh 1)$ or $\ln\left(2\left(\frac{e + e^{-1}}{2}\right)\right) = \ln\left(e + \frac{1}{e}\right)$	Obtains given answer from complete and correct work. Minimum for each route shown. Allow $\ln(e^{-1} + e)$	A1*
	Algebraic integration must be used		
	Note that there are potentially other ways e.g., factorising followed by log laws: $\ln\left(\frac{e^2 - e^{-2}}{2}\right) - \ln\left(\frac{e - e^{-1}}{2}\right) = \ln\left(\frac{1}{2}\left(e + \frac{1}{e}\right)\left(e - \frac{1}{e}\right)\right) - \ln\left(\frac{1}{2}\left(e - \frac{1}{e}\right)\right)$ M1 $= \ln\left(e + \frac{1}{e}\right) + \ln\left(\frac{1}{2}\left(e - \frac{1}{e}\right)\right) - \ln\left(\frac{1}{2}\left(e - \frac{1}{e}\right)\right)$ dM1 $= \ln\left(e + \frac{1}{e}\right)$ A1*		
			(4)
			Total 8

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td>8</td><td colspan="2">$I_n = \int_0^k x^n (k-x)^{\frac{1}{2}}\,dx \qquad n \geqslant 0$
If d(...) notation is used marks are only scored when it is removed.
Please see overleaf if the split is done first</td><td></td></tr>
<tr><td>(a)</td><td colspan="2">$u = x^n \qquad u' = nx^{n-1} \qquad v' = (k-x)^{\frac{1}{2}} \qquad v = -\frac{2}{3}(k-x)^{\frac{3}{2}}$
$I_n = \left[-\frac{2}{3}x^n (k-x)^{\frac{3}{2}}\right]_0^k - \int_0^k -\frac{2}{3}nx^{n-1}(k-x)^{\frac{3}{2}}\,dx$
M1: Uses parts in the correct direction to obtain an expression of the form
$\pm ...x^n (k-x)^{\frac{3}{2}} \pm \int ...x^{n-1}(k-x)^{\frac{3}{2}}(dx)$
A1: Correct expression (limits not required on either part and ‘dx’ may be missing)</td><td>M1
A1</td></tr>
<tr><td></td><td>$(I_n =)\ 0 + \frac{2}{3}n\int_0^k x^{n-1}(k-x)(k-x)^{\frac{1}{2}}\,dx$</td><td>Applies $(k-x)^{\frac{3}{2}} = (k-x)(k-x)^{\frac{1}{2}}$ to integral. Could be implied if work correct but do not accept going straight to "$\frac{2}{3}nkI_{n-1} - \frac{2}{3}nI_n$"
Requires previous M mark.</td><td>dM1</td></tr>
<tr><td></td><td>$\frac{2}{3}n\int_0^k \left(kx^{n-1}(k-x)^{\frac{1}{2}} - x^n (k-x)^{\frac{1}{2}}\right)dx$
$\Rightarrow \frac{2}{3}n(kI_{n-1} - I_n)$ or $\frac{2}{3}knI_{n-1} - \frac{2}{3}nI_n$ or
$\frac{2}{3}kn\int_0^k x^{n-1}(k-x)^{\frac{1}{2}}(dx) - \frac{2}{3}n\int_0^k x^n (k-x)^{\frac{1}{2}}(dx)$</td><td>Expands and writes RHS in terms of both I_n and I_{n-1} i.e., RHS = $..I_{n-1} \pm ...I_n$ with no other terms.
This mark is not available until the $\left[...x^n (k-x)^{\frac{3}{2}}\right]_0^k$ disappears.
Allow if actual integrals are used for both I_n and/or I_{n-1} and allow going straight to $\frac{2}{3}knI_{n-1} - \frac{2}{3}nI_n$ provided the split was seen.
Requires both previous M marks.</td><td>ddM1</td></tr>
<tr><td></td><td colspan="2">$\Rightarrow \left(1 + \frac{2}{3}n\right)I_n = \frac{2}{3}knI_{n-1}$ or $\Rightarrow \frac{3+2n}{3}I_n = \frac{2}{3}knI_{n-1}$
$\Rightarrow I_n = \frac{2kn}{3+2n}I_{n-1}$ *
Reaches given answer with no mathematical errors seen. Allow poor bracketing if it is recovered. There must be at least one non-trivial intermediate line where the LHS $= \mathrm{f}(n)I_n$ allowing e.g., $I_n + \frac{2}{3}I_n = ...$
Allow minor variations in given answer e.g., $I_n = \frac{2nkI_{n-1}}{2n+3}$
Condone missing ‘dx’s and allow if limits only seen once but $\left[-\frac{2}{3}x^n (k-x)^{\frac{3}{2}}\right]_0^k$
must be replaced by “0” or better</td><td>A1*</td></tr>
<tr><td></td><td colspan="2"></td><td>(5)</td></tr>
</table>

Question Number	Scheme/Notes	Marks
8(a) **Alt** **Split first**	$I_n = \int_0^k x^n (k-x)^{\frac{1}{2}}\,dx = \int_0^k x^n (k-x)(k-x)^{-\frac{1}{2}}\,dx = \int_0^k kx^n (k-x)^{-\frac{1}{2}}\,dx - \int_0^k x^{n+1}(k-x)^{-\frac{1}{2}}\,dx$ $= \left[-2kx^n (k-x)^{\frac{1}{2}}\right]_0^k + \int_0^k 2knx^{n-1}(k-x)^{\frac{1}{2}}\,dx + \left[2x^{n+1}(k-x)^{\frac{1}{2}}\right]_0^k - \int_0^k 2(n+1)x^n (k-x)^{\frac{1}{2}}\,dx$ $\Rightarrow 0 + 2knI_{n-1} + 0 - 2(n+1)I_n \Rightarrow (3+2n)I_n = 2knI_{n-1} \Rightarrow I_n = \frac{2kn}{3+2n}I_{n-1}$ * For attempts like this award the first 2 method marks **together** for applying the split, expanding **and** applying parts to achieve a correct form. The first accuracy mark can be awarded for a correct expression (limits not required on either part and 'dx's may be missing). As main scheme for the following two marks (note that in this case the first and third terms must both be replaced by "0" or better). There is no mark for just applying the split.	**(5)**

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td rowspan="6">8(b)</td><td colspan="2">$\int_0^k x^2(k-x)^{\frac{1}{2}}\,dx=\frac{9\sqrt{3}}{280}$ $I_n=\frac{2kn}{3+2n}I_{n-1}$</td><td></td></tr>
<tr><td>$I_2=\frac{4k}{7}I_1=\frac{4k}{7}\left(\frac{2k}{5}I_0\right)$
or $I_2=\frac{4k}{7}I_1,\quad I_1=\frac{2k}{5}I_0$</td><td>Attempts I_2 in terms of I_0 or
I_2 in terms of I_1 and I_1 in terms of I_0
Accept with their I_0 substituted
if I_0 attempted first. Allow $I_0=1$ to be used
(i.e., I_0 lost)
See note below if only see I_2 in terms of I_1</td><td>M1</td></tr>
<tr><td>$I_0=\int_0^k (k-x)^{\frac{1}{2}}\,dx=\left[-\frac{2}{3}(k-x)^{\frac{3}{2}}\right]_0^k$</td><td>$I_0=\ldots(k-x)^{\frac{3}{2}}$
Limits do not have to be seen or applied</td><td>M1</td></tr>
<tr><td colspan="2">$I_2=\frac{8k^2}{35}\times\frac{2}{3}k^{\frac{3}{2}}\Rightarrow\frac{16}{105}k^{\frac{7}{2}}=\frac{9\sqrt{3}}{280}\Rightarrow k=\ldots$
Solves an equation of the form $\frac{a}{b}k^{\frac{c}{2}}=\frac{9\sqrt{3}}{280}$ where $a,\ b\in\mathbb{Z}^+$, $\frac{a}{b}\notin\mathbb{Z}$, $c=5$ or 7
and where the LHS is their I_2. No processing or working requirements just look for a <u>value or numerical expression</u> for k from an appropriate equation.
May see $k=e^{\frac{2}{7}\ln\left(\frac{27\sqrt{3}}{128}\right)}$ or other logarithmic work.
Requires both previous M marks.
Note that $\frac{16}{105}k^{\frac{5}{2}}=\frac{9\sqrt{3}}{280}\Rightarrow k=\sqrt[5]{\frac{2187}{16384}}$ or 0.668...</td><td>ddM1</td></tr>
<tr><td>$k^{\frac{7}{2}}=\frac{27\sqrt{3}}{128}\Rightarrow k^7=\frac{2187}{16384}\Rightarrow k=\frac{3}{4}$</td><td>Correct exact <u>value</u> for k <u>from a correct equation</u>.
Not $\sqrt[7]{\frac{2187}{16384}}$ nor $\pm\frac{3}{4}$</td><td>A1</td></tr>
<tr><td colspan="2">Note that if I_2 is only found in terms of I_1 then award the first two marks together when a correct form for I_1 is achieved i.e.,
$\ldots x(k-x)^{\frac{3}{2}}+\ldots(k-x)^{\frac{5}{2}}$ or $\ldots(\ldots x+\ldots k)(k-x)^{\frac{3}{2}}$
Using parts:
$I_1=\left[-\frac{2}{3}x(k-x)^{\frac{3}{2}}-\frac{4}{15}(k-x)^{\frac{5}{2}}\right]_0^k=\frac{4}{15}k^{\frac{5}{2}}$
Using substitution:
$u=k-x\ \Rightarrow I_1=\int_0^k x(k-x)^{\frac{1}{2}}\,dx=\left[-\frac{2}{15}(3x+2k)(k-x)^{\frac{3}{2}}\right]_0^k=\frac{4}{15}k^{\frac{5}{2}}$
There are no marks if the reduction formula is not used including direct attempts at I_2 or if $k=\frac{3}{4}$ is arrived at by purely solving the integral equation on a calculator</td><td></td></tr>
<tr><td colspan="4" align="right">(4)</td></tr>
<tr><td colspan="4" align="right">Total 9</td></tr>
</table>

Question Number	Scheme	Notes	Marks
9	May use **i**, **j**, **k** notation		
9(a)	$\mathbf{n} = \begin{pmatrix}3\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\-2\\2\end{pmatrix} = \ldots \quad \left\{\begin{pmatrix}2\\-5\\-6\end{pmatrix}\right\}$	Calculates the vector product of two vectors in Π_1 (two components correct)	**M1**
	$\begin{pmatrix}5\\3\\0\end{pmatrix} . \begin{pmatrix}2\\-5\\-6\end{pmatrix} = \ldots \quad \{-5\}$	Calculates the scalar product of a point in the plane and their normal. Not dependent but must follow an attempt at a vector product which could be poor, e.g., $3\mathbf{i}+2\mathbf{k}$. Value must be correct if there is no indication of a correct method to evaluate the scalar product.	**M1**
	$\mathbf{r}.\begin{pmatrix}2\\-5\\-6\end{pmatrix} = \begin{pmatrix}5\\3\\0\end{pmatrix}.\begin{pmatrix}2\\-5\\-6\end{pmatrix} \Rightarrow 2x-5y-6z=-5$	Any correct Cartesian equation, e.g., $-2x+5y+6z=5$ $2x-5y-6z+5=0$	**A1**
			(3)
Alt **Sim eqns**	$x=5+3s+t$ $y=3-2t \quad \Rightarrow$ e.g., $y+z=3+s$ $z=s+2t$	Forms simultaneous equations in x, y, z, s and t and obtains an equation that eliminates at least one of s and t	**M1**
	$x=5+3(y+z-3)+\frac{1}{2}z-\frac{1}{2}(y+z-3)$ $x=\frac{5}{2}y+3z-\frac{5}{2}$	M1: Proceeds to an equation in x, y and z only A1: Any correct equation with terms collected	**M1** **A1**
			(3)

Question Number	Scheme	Notes	Marks
9(b) **Way 1**	$2x-5y-6z=-5, \quad 5x-2y+3z=1$ $\Rightarrow$ e.g., $12x-9y=-3$	Uses both plane equations to eliminate one variable. May see $21y+36z=27,\ 21x+27z=15$	**M1**
	e.g., $4x-3y=-1 \Rightarrow x=\dfrac{3y-1}{4} \Rightarrow y=\dfrac{4x+1}{3}$ $3z=1-\dfrac{5(3y-1)}{4}+2y=\dfrac{4-15y+5+8y}{4} \Rightarrow z=\dfrac{9-7y}{12} \Rightarrow y=\dfrac{12z-9}{-7}$ Expresses one variable in terms of the other two (single underlining) or expresses two variables in terms of the other one (double underlining). This work may be seen by setting a variable equal to a parameter to find the other variables in terms of the parameter (or the parameter in terms of the other two variables) e.g., $y=\lambda,\quad x=f(\lambda),\quad z=g(\lambda) \quad \left\{\Rightarrow x=\dfrac{-1+3\lambda}{4},\ y=\lambda,\ z=\dfrac{9-7\lambda}{12}\right\}$ $y=\lambda,\quad \lambda=f(x),\quad \lambda=g(z) \quad \left\{\Rightarrow \lambda=\dfrac{4x+1}{3},\ y=\lambda,\ \lambda=\dfrac{12z-9}{-7}\right\}$ See examples below. **Requires previous M mark.**		**dM1**
	e.g., $\dfrac{4x+1}{3}=y=\dfrac{12z-9}{-7} \Rightarrow \dfrac{x+\frac{1}{4}}{\frac{3}{4}}=\dfrac{y-0}{1}=\dfrac{z-\frac{3}{4}}{-\frac{7}{12}}$ or e.g., $x=\dfrac{-1+3\lambda}{4},\ y=\lambda,\ z=\dfrac{9-7\lambda}{12} \Rightarrow$ $\Rightarrow \mathbf{r}=\begin{pmatrix}-\frac{1}{4}\\0\\\frac{3}{4}\end{pmatrix}+\lambda\begin{pmatrix}\frac{3}{4}\\1\\-\frac{7}{12}\end{pmatrix}$ or e.g. $\mathbf{r}=\begin{pmatrix}-\frac{1}{4}\\0\\\frac{3}{4}\end{pmatrix}+\lambda\begin{pmatrix}9\\12\\-7\end{pmatrix}$	**dd**M1: Correct method to form RHS of vector equation. Allow slips but must not be a clearly incorrect method (e.g., point and direction confused, all non-zero point coordinates the wrong sign, no attempt seen or implied to obtain single coefficients for the variables in the numerator where necessary). Allow this mark if the point is later changed by multiplication e.g., $\left(-\frac{1}{4},\ 0,\ \frac{3}{4}\right)$ becomes $(-1,\ 0,\ 3)$ Condone missing $\mathbf{r}$ =... Allow this mark if $(\mathbf{r}-\mathbf{a})\times\mathbf{b}(=0)$ or $\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$ are appropriately used. **Requires both previous M marks.** A1: Any correct **equation** (with any parameter). Do not condone e.g., $l=$ ... Do not isw if the point is changed by multiplication.	**ddM1** **A1**
examples	$x=\dfrac{3y-1}{4}=\dfrac{5-9z}{7} \Rightarrow \dfrac{x-0}{1}=\dfrac{y-\frac{1}{3}}{\frac{4}{3}}=\dfrac{z-\frac{5}{9}}{-\frac{7}{9}}$ or $x=\lambda,\ y=\dfrac{4\lambda+1}{3},\ z=\dfrac{5-7\lambda}{9} \Rightarrow \mathbf{r}=\begin{pmatrix}0\\\frac{1}{3}\\\frac{5}{9}\end{pmatrix}+\lambda\begin{pmatrix}1\\\frac{4}{3}\\-\frac{7}{9}\end{pmatrix}$ $\dfrac{5-7x}{9}=\dfrac{9-7y}{12}=z \Rightarrow \dfrac{x-\frac{5}{7}}{-\frac{9}{7}}=\dfrac{y-\frac{9}{7}}{-\frac{12}{7}}=\dfrac{z-0}{1}$ or $x=\dfrac{5-9\lambda}{7},\ y=\dfrac{12z-9}{-7},\ z=\lambda \Rightarrow \mathbf{r}=\begin{pmatrix}\frac{5}{7}\\\frac{9}{7}\\0\end{pmatrix}+\lambda\begin{pmatrix}-\frac{9}{7}\\-\frac{12}{7}\\1\end{pmatrix}$		**(4)**

Question Number	Scheme	Notes	Marks
9(b) **Way 2** **Finds point and takes vector product of normals**	Work may be minimal if they obtain a correct point. But do not accept just sight of an incorrect point without some evidence of an appropriate method to obtain it.		
	$2x-5y-6z=-5, \quad 5x-2y+3z=1$ Let $y=0 \Rightarrow 2x-6z=-5, \quad 5x+3z=1$ or $\Rightarrow$ e.g., $12x-9y=-3$	Assigns a value to one variable to obtain two equations in the other variables or eliminates one variable as in Way 1.	**M1**
	$\Rightarrow 12x=-3 \Rightarrow x=-\frac{1}{4},\ y=0,\ z=\frac{3}{4}$ May see $\left(0, \frac{1}{3}, \frac{5}{9}\right)$ or $\left(\frac{5}{7}, \frac{9}{7}, 0\right)$	Solves or assigns a value to one variable to find values for the other variables. There is no need to check a point that arises from no working provided it is clear that the previous M mark has been scored. **Requires previous M mark.**	**dM1**
	Note that a point could be obtained via substituting the given form of Π_1 into Π_2 and expanding (M1) and then finding values of *s* and *t* that satisfy the equation and then finding a point (dM1)		
	$\begin{pmatrix}2\\-5\\-6\end{pmatrix}\times\begin{pmatrix}5\\-2\\3\end{pmatrix}=\begin{pmatrix}-27\\-36\\21\end{pmatrix}\Rightarrow\begin{pmatrix}-\frac{1}{4}\\0\\\frac{3}{4}\end{pmatrix}+\lambda\begin{pmatrix}-27\\-36\\21\end{pmatrix}$	Calculates vector product of normals (two components correct) **and** forms RHS of vector equation (allowing for copying slips but must not confuse point and direction). Allow this mark if the point is later changed by multiplication. Condone missing $\mathbf{r}=$... Allow this mark if $(\mathbf{r}-\mathbf{a})\times\mathbf{b}(=0)$ or $\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$ are appropriately used. **Requires both previous M marks.**	**ddM1**
	$\Rightarrow \mathbf{r}=\begin{pmatrix}-\frac{1}{4}\\0\\\frac{3}{4}\end{pmatrix}+\lambda\begin{pmatrix}-27\\-36\\21\end{pmatrix}$ or e.g., $\mathbf{r}=\begin{pmatrix}-\frac{1}{4}\\0\\\frac{3}{4}\end{pmatrix}+\lambda\begin{pmatrix}-9\\-12\\7\end{pmatrix}$	Any correct **equation** in this form (with any parameter). Do not condone e.g., $l=$ ... Do not isw if the point is changed by multiplication. Correct points will have the form $\left(\frac{3\alpha-1}{4}, \alpha, \frac{9-7\alpha}{12}\right)$	**A1**
			(4)
Way 3 **2 points**	Finding 2 points on the line and **subtract** for direction e.g., Finds $\left(-\frac{1}{4}, 0, \frac{3}{4}\right)$ (M1dM1 as Way 2) Then finds $\left(0, \frac{1}{3}, \frac{5}{9}\right) \Rightarrow$ direction $=\left(\frac{1}{4}, \frac{1}{3}, -\frac{7}{36}\right) \Rightarrow$ forms RHS of vector equation (ddM1) Then A1 for a correct equation		
			(4)
	Correct points/positions include: $\begin{pmatrix}0\\\frac{1}{3}\\\frac{5}{9}\end{pmatrix}\ \begin{pmatrix}-\frac{1}{4}\\0\\\frac{3}{4}\end{pmatrix}\ \begin{pmatrix}\frac{5}{7}\\\frac{9}{7}\\0\end{pmatrix}\ \begin{pmatrix}1\\\frac{5}{3}\\-\frac{2}{9}\end{pmatrix}\ \begin{pmatrix}\frac{1}{2}\\1\\\frac{1}{6}\end{pmatrix}\ \begin{pmatrix}-\frac{4}{7}\\-\frac{3}{7}\\1\end{pmatrix}\ \begin{pmatrix}-1\\-1\\\frac{4}{3}\end{pmatrix}\ \begin{pmatrix}2\\3\\-1\end{pmatrix}$		

Question Number	Scheme	Notes	Marks
9(c)	Note that use of their line from part (b) must be seen to score any marks in (c)		
	$\mathbf{r}=\begin{pmatrix}-\frac{1}{4}\\0\\\frac{3}{4}\end{pmatrix}+\lambda\begin{pmatrix}9\\12\\-7\end{pmatrix}=\begin{pmatrix}-\frac{1}{4}+9\lambda\\12\lambda\\\frac{3}{4}-7\lambda\end{pmatrix}$ $4\left(-\frac{1}{4}+9\lambda\right)-3(12\lambda)-\left(\frac{3}{4}-7\lambda\right)=0\Rightarrow 7\lambda=\frac{7}{4}\Rightarrow\lambda=\frac{1}{4}$	Substitutes the parametric form of their line (allow slips but must not clearly confuse position and direction) from (b) into Π_3 and solves for λ The "=0" could be implied by a solution.	**M1**
	$\Rightarrow\left(9\left(\frac{1}{4}\right)-\frac{1}{4},\ 12\left(\frac{1}{4}\right),\ -7\left(\frac{1}{4}\right)+\frac{3}{4}\right)=\ldots$ Substitutes their λ into their line and obtains a point/position vector with values for all coordinates/components. If there is no working at least two coordinates/components should be consistent with their equation or parametric form. Isw if the point/position is altered by multiplication. **Requires previous M mark.**		**dM1**
	$(2,\ 3,\ -1)$	Correct point. No others. Allow $x=\ldots,\ y=\ldots,\ z=\ldots$ and condone as a position vector. Do not isw.	**A1**
			(3)
			Total 10
			PAPER TOTAL 75