January 2024
WFM02 Further Pure Mathematics F2 Mark Scheme

Question Number	Scheme	Notes	Marks
1	$\dfrac{1}{x+2} > 2x+3$		
	Examples: $\dfrac{1-(x+2)(2x+3)}{x+2} > 0 \Rightarrow 2x^2+7x+5=0$ $x+2 > (2x+3)(x+2)^2$ $\Rightarrow (x+2)\left(2x^2+7x+5\right)=0$ or $2x^3+11x^2+19x+10=0$ $\dfrac{1}{x+2}=2x+3 \Rightarrow (2x+3)(x+2)-1=2x^2+7x+5=0$ Uses algebra to obtain a 3TQ, $(x+2)$ multiplied by a 3TQ or a 4TC. Allow slips and condone incorrect inequality signs but the first algebraic step should be otherwise appropriate so do not accept work with e.g., $(2x+3)(x+2)=0$. The "= 0" can be implied by solutions. Graphical attempts require intersections to be found algebraically. Squaring first is acceptable so allow M1 for obtaining a 5TQ $(4x^4+28x^3+73x^2+84x+35=0)$		**M1**
	e.g., $(2x+5)(x+1)=0 \Rightarrow$ $x=-\dfrac{5}{2}, -1$	Both -1 and $-\frac{5}{2}$ from appropriate work and no extra incorrect cvs. May only be seen in the solution set. Allow solving a 3TQ etc. by calculator.	**A1**
	$x=-2$	Identifies -2 as a critical value. May only be seen in solution set. This is the only mark available if there is no algebraic manipulation seen. Allow from any or no working e.g., from $(2x+3)(x+2)=0$	**B1**
	$\Rightarrow x < -\dfrac{5}{2},\ -2 < x < -1$ or e.g., $(-\infty, -2.5), (-2, -1)$ **M1**: For the regions $x < a,\ -2 < x < b$ with real cvs $a < -2$ and $b > -2$ **but condone** $b < x < -2$ as a notational slip for this mark. Condone any non-strict inequality signs and poor notation for this mark. Not dependent but must follow an attempt at algebraic manipulation. **A1**: Correct solution set in any form. Do not isw if the correct inequalities are subsequently incorrectly amended. Allow all marks even if an incorrect inequality sign was seen earlier in the working.		**M1 A1**
	Examples: $-\dfrac{5}{2} > x$ or $-2 < x < -1$ M1 A1 $\qquad x < -\dfrac{5}{2}$ and $-2 < x < -1$ M1 A1 (Accept any word between the two correct regions) $x < -\frac{5}{2},\ -1 < x < -2$ M1 A0 (notational slip) $\left(-\infty, -\dfrac{5}{2}\right) \cap (-2, -1)$ M1A0 (incorrect symbol – allow "and") $\left[-\infty, -\dfrac{5}{2}\right] \cup [-2, -1]$ M1A0 $x < -\frac{5}{2} \quad -2 < x \quad x < -1$ M0 A0 (insufficient)		
			(5)
			Total 5

Question Number	Scheme	Notes	Marks
2(a)	**(i)** $z=6-6\sqrt{3}\mathrm{i} \Rightarrow \lvert z\rvert=\sqrt{6^2+\left(6\sqrt{3}\right)^2}=12$	+12 only. Accept if just stated	**B1**
	(ii) e.g., $\arg z=-\arctan\dfrac{6\sqrt{3}}{6}$ Attempts an expression for a relevant angle. Look for $\pm\arctan\left(\pm\dfrac{6\sqrt{3}}{6}\right)$ or e.g., $\pm\tan^{-1}\left(\pm\dfrac{1}{\sqrt{3}}\right)$ If arctan is not seen allow e.g., $\tan\alpha=\dfrac{6\sqrt{3}}{6}\Rightarrow\alpha=\dfrac{\pi}{3}$ with α correct for their $\tan\alpha$ If using sin or cos the hypotenuse must be their 12		**M1**
	$\arg z$ or arg or argument (of z) $=-\dfrac{\pi}{3}$ * A correct proof with no incorrect work/statements. LHS required. **Allow "$\theta=$" if consistent**, e.g., $\theta=-\frac{\pi}{3}$ cannot follow "$\tan\theta=+\sqrt{3}$"		**A1***
(ii) Way 2	$z=12\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)=12\left(\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ or $12\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}$ or $\cos\theta=\frac{1}{2}$ or $\sin\theta=-\frac{\sqrt{3}}{2}$ [M1] $\Rightarrow\arg z=-\frac{\pi}{3}$ [A1*] M1: Factorises out 12 and writes in trig or exp form or identifies $\cos\theta=\frac{1}{2}$ and $\sin\theta=-\frac{\sqrt{3}}{2}$ A1: Acceptable statement with all work correct		
(ii) Way 3	$z=12\left(\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ or $12\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}$ or $12\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)=6-6\sqrt{3}\mathrm{i}$ [M1] $\Rightarrow\arg z=-\frac{\pi}{3}$ [A1*] M1: Assumes result, writes correctly for their 12 and attempts $a+\mathrm{i}b$ form A1: Obtains $6-6\sqrt{3}\mathrm{i}$ and makes acceptable statement with all work correct		
			(3)
(b)	$z=\text{"12"}\left(\cos\left(-\dfrac{\pi}{3}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{3}\right)\right)$ or $\text{"12"}e^{-\frac{\pi}{3}\mathrm{i}}$ [no missing "i" unless recovered] Correct trig or exp. form with their 12. Could be implied by their z^4 in trig or exp. form e.g., $(\text{"12"}e^{-\frac{\pi}{3}\mathrm{i}})^4$ Allow equivalent values of θ e.g. $\frac{5\pi}{3}$ and use of e.g., $\sin\left(-\frac{\pi}{3}\right)=-\sin\left(\frac{\pi}{3}\right)$. Condone poor bracketing. Allow this mark if $+2k\pi,-2k\pi,\pm2k\pi$ appears with argument		**M1**
	$z^4=20736\left(\cos\left(-\dfrac{4\pi}{3}\right)+\mathrm{i}\sin\left(-\dfrac{4\pi}{3}\right)\right)$ or $20736\left(\cos-\dfrac{4\pi}{3}+\mathrm{i}\sin-\dfrac{4\pi}{3}\right)$ or $20736e^{-\frac{4\pi}{3}\mathrm{i}}$ Correct z^4 in any form. 12^4 evaluated and arg. of $-\frac{4\pi}{3}$ (not just $4\times-\frac{\pi}{3}$) or $\frac{2\pi}{3}$ only although may use e.g., $\sin\left(-\frac{4\pi}{3}\right)=-\sin\left(\frac{4\pi}{3}\right)$. No "$k$"s. Condone an "unclosed" bracket. Only accept $-10368+10368\sqrt{3}\mathrm{i}$ or $20736\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)$ **provided evidence of de Moivre.**		**A1**
			(2)

Question Number	Scheme	Notes	Marks
2(c)	$w=z^{\frac{1}{2}}=(\pm)\sqrt{"12"}\left(\cos\left(\frac{-\frac{\pi}{3}}{2}\right)+\text{i}\sin\left(\frac{-\frac{\pi}{3}}{2}\right)\right)$ or e.g., $(\pm)"2\sqrt{3}"e^{-\frac{\pi}{6}\text{i}}$ [no missing "i" unless recovered] Correct use of de Moivre's theorem with $-\frac{\pi}{3}$ and their 12 to attempt one square root. Allow work with argument of $\frac{5\pi}{3}$ for $-\frac{\pi}{3}$ and use of e.g., $\sin\left(-\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)$. Condone poor bracketing. M0 if z^4 used for z. Allow this mark if $+2k\pi,-2k\pi,\pm2k\pi$ appears with argument		M1
	$w=3-\sqrt{3}\text{i},-3+\sqrt{3}\text{i}$ oe A1ft: One correct exact root in a +ib or c(a +ib) form (a, b, c may be unsimplified but not numerical trig expressions) **ft their 12 only** i.e.$(\pm)\sqrt{"12"}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\text{i}\right)$ A1: Both exact roots (no others) correct in a +ib form – a and b may be unsimplified (but not numerical trig expressions) e.g. accept $a=(\pm)\sqrt{12}\frac{\sqrt{3}}{2},(\pm)\frac{\sqrt{36}}{2}\quad b=(\mp)\frac{\sqrt{12}}{2},(\mp)\frac{2\sqrt{3}}{2}$ Accept $\pm\left(3-\sqrt{3}\text{i}\right)$ but just $\pm3-\sqrt{3}\text{i}$ is A1 A0. Just $\pm\sqrt{3}\left(\sqrt{3}-\text{i}\right)$ is A1 A0		A1ft A1
	Note: $w^2=r^2\left(\cos2\theta+i\sin2\theta\right)=z\Rightarrow r,\ \theta,\ w=...$ is an acceptable approach		(3)
Alt	$w^2=z\Rightarrow(a+\text{i}b)^2=a^2-b^2+2ab\text{i}=6-6\sqrt{3}\text{i}\Rightarrow a^2-b^2=6,\ 2ab=-6\sqrt{3}$ $b=-\frac{3\sqrt{3}}{a}\Rightarrow a^2-\frac{27}{a^2}=6\Rightarrow a^4-6a^2-27=\left(a^2-9\right)\left(a^2+3\right)=0\Rightarrow a^2=9,\ a=\pm3,\ b=\mp\sqrt{3}$ **M1**: From a correct starting point, expands and equates real and imaginary parts to form two equations in a and b and obtains at least one value for both a and b $w=3-\sqrt{3}\text{i},-3+\sqrt{3}\text{i}$ **A1**: One correct exact root in a +ib or c(a +ib) form (a, b, c may be unsimplified) **A1**: Both exact roots (no others) correct in a +ib form – a and b may be unsimplified		
			Total 8

Question Number	Scheme	Notes	Marks
3(a)	$\dfrac{r}{\sqrt{r(r+1)}+\sqrt{r(r-1)}}\times\dfrac{\sqrt{r(r+1)}-\sqrt{r(r-1)}}{\sqrt{r(r+1)}-\sqrt{r(r-1)}}$	A correct multiplier to rationalise the denominator seen or implied by correct work	**M1**
	$=\dfrac{r\left(\sqrt{r(r+1)}-\sqrt{r(r-1)}\right)}{r(r+1)-r(r-1)}=\dfrac{\sqrt{r(r+1)}-\sqrt{r(r-1)}}{2}$ or $A=\dfrac{1}{2}$ Correct expression or correct value for A. Condone poor notation if intention clear. There must be (minimal) correct supporting working.		**A1**
	Alternative: $A=\dfrac{r}{\left(\sqrt{r(r+1)}+\sqrt{r(r-1)}\right)\left(\sqrt{r(r+1)}-\sqrt{r(r-1)}\right)}=\dfrac{r}{r(r+1)-r(r-1)}$ or $\dfrac{r}{r^2+r-r^2+r}$ or $\dfrac{r}{2r}\Rightarrow A=\dfrac{1}{2}$ M1: Correctly makes A the subject A1: Correct completion with one intermediate fraction		
			(2)
(b)	$\displaystyle\sum_{r=1}^{n}\frac{r}{\sqrt{r(r+1)}+\sqrt{r(r-1)}}="\frac{1}{2}"\left(\begin{array}{c}\sqrt{1\times2}-\sqrt{1\times0}\ \left(=\sqrt{2}\ (-0)\right)\\ +\sqrt{2\times3}-\sqrt{2\times1}\ \left(=\sqrt{6}-\sqrt{2}\right)+\ldots\\ \ldots+\sqrt{(n-1)(n-1+1)}-\sqrt{(n-1)(n-1-1)}\ \left(=\sqrt{n(n-1)}-\sqrt{(n-1)(n-2)}\right)\\ +\sqrt{n(n+1)}-\sqrt{n(n-1)}\end{array}\right)$ **M1**: Applies the method of differences for $r=1$ and $r=n$ in the given expression with or without their A and obtains one correct row of these 2. **M1**: Applies the method of differences for $r=1$, $r=n$ and **either** $r=2$ or $r=n-1$ in the given expression with/without their A and obtains 2 correct rows of these 4. When considering how many rows are correct, if A has been clearly applied to any term then assess all rows as if A has been applied throughout. Condone missing bracket if their A is applied to a row e.g., "$\frac{1}{2}\times\ \sqrt{6}-\sqrt{2}$" <u>if it is recovered</u> but e.g., $\frac{\sqrt{6}}{2}-\sqrt{2}$ is an incorrect row. Ignore a row for $r=0$. Condone equivalent work with r or e.g., k used for n. **Both marks can be implied by a correct final expression with or without their *A* provided there are at least any two correct rows of differences** i.e., $"\frac{1}{2}"\left(\sqrt{n(n+1)}-0\right)$ or $\sqrt{n(n+1)}-0$		**M1** **M1**
	Note: row 3 is $"\frac{1}{2}"(\sqrt{12}\,(\text{or }2\sqrt{3})-\sqrt{6})$, row 4 is $"\frac{1}{2}"(\sqrt{20}\,(\text{or }2\sqrt{5})-\sqrt{12}\,(\text{or }2\sqrt{3}))$ If $\frac{1}{2}$ is fully applied the rows are: $\frac{\sqrt{2}}{2},\ \frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2},\ \frac{\sqrt{12}}{2}(\text{or }\sqrt{3})-\frac{\sqrt{6}}{2},\ \frac{\sqrt{20}}{2}(\text{or }\sqrt{5})-\frac{\sqrt{12}}{2}(\text{or }\sqrt{3}),\ldots$ $\ldots,\frac{\sqrt{(n-2)(n-1)}}{2}-\frac{\sqrt{(n-2)(n-3)}}{2},\ \frac{\sqrt{n(n-1)}}{2}-\frac{\sqrt{(n-1)(n-2)}}{2},\ \frac{\sqrt{n(n+1)}}{2}-\frac{\sqrt{n(n-1)}}{2}$		
	$=\dfrac{1}{2}\sqrt{n(n+1)}$ oe e.g., $\dfrac{\sqrt{n^2+n}}{2}$	Correct expression in terms of n. No incorrect terms seen in differences work even if cancelled but condone the occasional poor bracket. There should be no "0" so e.g., $\frac{1}{2}\left(\sqrt{n(n+1)}-0\right)$ is A0 Does not require marks in (a)	**A1**
			(3)

Question Number	Scheme	Notes	Marks
3(c)	$\sum r = \frac{1}{2}n(n+1)$ e.g., sight of $k \times ... = \sqrt{\frac{1}{2}n(n+1)}$	States or uses the correct summation formula for integers	**M1**
	$\frac{k}{2}\sqrt{n(n+1)} = \sqrt{\frac{1}{2}n(n+1)} \Rightarrow \frac{k}{2} = \sqrt{\frac{1}{2}} \Rightarrow k = \sqrt{2}$	$\sqrt{2}$ only (Not $\pm$). $k = \sqrt{2}$ must not come from a clearly incorrect equation.	**A1**
			(2)
			Total 7

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td rowspan="7">4(a)</td><td>$y=\tan\left(\frac{3x}{2}\right)\Rightarrow y'=\frac{3}{2}\sec^2\left(\frac{3x}{2}\right)$</td><td>Any correct first derivative. Not implied by $y'\left(\frac{\pi}{6}\right)=3$</td><td>B1</td></tr>
<tr><td>$\Rightarrow y''=2\times\frac{3}{2}\sec\left(\frac{3x}{2}\right)\times\sec\left(\frac{3x}{2}\right)\tan\left(\frac{3x}{2}\right)\times\frac{3}{2}$
$\left[=\frac{9}{2}\sec^2\left(\frac{3x}{2}\right)\tan\left(\frac{3x}{2}\right)\right]$</td><td>Attempts the second derivative achieving $k\sec^2\left(\frac{3x}{2}\right)\tan\left(\frac{3x}{2}\right)$ or unsimplified equivalent. Not implied by $y''\left(\frac{\pi}{6}\right)=9$</td><td>M1</td></tr>
<tr><td>$\Rightarrow y'''=\frac{9}{2}\sec^2\left(\frac{3x}{2}\right)\sec^2\left(\frac{3x}{2}\right)\times\frac{3}{2}+\frac{9}{2}\tan\left(\frac{3x}{2}\right)\times 2\times\frac{3}{2}\sec^2\left(\frac{3x}{2}\right)\tan\left(\frac{3x}{2}\right)$
$\left[=\frac{27}{4}\sec^4\left(\frac{3x}{2}\right)+\frac{27}{2}\sec^2\left(\frac{3x}{2}\right)\tan^2\left(\frac{3x}{2}\right)\right]$</td><td>dM1: Attempts third derivative using the product rule, achieving $P\sec^4\left(\frac{3x}{2}\right)+Q\sec^2\left(\frac{3x}{2}\right)\tan^2\left(\frac{3x}{2}\right)$ or unsimplified equivalent. Requires previous M mark.
A1: Correct differentiation. Accept unsimplified. Not implied by $y'''\left(\frac{\pi}{6}\right)=54$</td><td>dM1
A1</td></tr>
<tr><td colspan="2">If $\sec^2\left(\frac{3x}{2}\right)=\tan^2\left(\frac{3x}{2}\right)+1$ is used the identity must be used correctly and to score M marks expressions of consistent form should be achieved.
Note that replacing $\sec^2\left(\frac{3x}{2}\right)$ in $y''\Rightarrow y'''=\frac{27}{4}\sec^2\left(\frac{3x}{2}\right)+\frac{81}{4}\sec^2\left(\frac{3x}{2}\right)\tan^2\left(\frac{3x}{2}\right)$</td><td></td></tr>
<tr><td colspan="2">$y\left(\frac{\pi}{6}\right)=1, y'\left(\frac{\pi}{6}\right)=3, y''\left(\frac{\pi}{6}\right)=9, y'''\left(\frac{\pi}{6}\right)=54$
Attempts values (but allow numerical trig expressions) for y and their 3 derivatives at $\frac{\pi}{6}$ - accept stated values or insertion into a series of the correct form</td><td>M1</td></tr>
<tr><td colspan="2">$(y=)1+3\left(x-\frac{\pi}{6}\right)+\frac{9}{2!}\left(x-\frac{\pi}{6}\right)^2+\frac{54}{3!}\left(x-\frac{\pi}{6}\right)^3+\ldots$
Applies Taylor's correctly about $\frac{\pi}{6}$ with their values/numerical trig expressions. If values are not seen separately the work should imply a correct formula but allow a recognisable attempt at the series following the correct general formula stated. Requires previous M mark.</td><td>dM1</td></tr>
<tr><td>$(y=)1+3\left(x-\frac{\pi}{6}\right)+\frac{9}{2}\left(x-\frac{\pi}{6}\right)^2+9\left(x-\frac{\pi}{6}\right)^3+\ldots$</td><td>Correct expression with coeffs. in simplest form. "$y=\ldots$" not required. Requires all previous marks.
Score A0 if clear evidence of <u>use</u> of any wrong derivative expression.</td><td>A1</td></tr>
<tr><td colspan="3">If e.g. $y'''\left(\frac{\pi}{6}\right)$ is found by calculator but $y'(x)$ and $y''(x)$ were seen award 1100110 max</td><td>(7)</td></tr>
<tr><td></td><td colspan="2">Note: With responses that work in sin and cos throughout, to score M marks there must be no loss of form when differentiating (sign and coefficient errors only, also allowing sign errors with product/quotient formulae). Any use of identities must be correct. E.g:
$y=\tan\left(\frac{3x}{2}\right)=\frac{\sin\left(\frac{3x}{2}\right)}{\cos\left(\frac{3x}{2}\right)}\Rightarrow y'=\frac{\frac{3}{2}\cos^2\left(\frac{3x}{2}\right)+\frac{3}{2}\sin^2\left(\frac{3x}{2}\right)}{\cos^2\left(\frac{3x}{2}\right)}$
$y''=\frac{\frac{9}{2}\cos^3\left(\frac{3x}{2}\right)\sin\left(\frac{3x}{2}\right)+\frac{9}{2}\cos\left(\frac{3x}{2}\right)\sin^3\left(\frac{3x}{2}\right)}{\cos^4\left(\frac{3x}{2}\right)}$ or $\frac{\frac{9}{2}\cos\left(\frac{3x}{2}\right)\sin\left(\frac{3x}{2}\right)}{\cos^4\left(\frac{3x}{2}\right)}$ or $\frac{9\sin\left(\frac{3x}{2}\right)}{2\cos^3\left(\frac{3x}{2}\right)}$
$y'''=\frac{\frac{27}{4}\cos^8\left(\frac{3x}{2}\right)+27\cos^6\left(\frac{3x}{2}\right)\sin^2\left(\frac{3x}{2}\right)+\frac{81}{4}\cos^4\left(\frac{3x}{2}\right)\sin^4\left(\frac{3x}{2}\right)}{\cos^8\left(\frac{3x}{2}\right)}=\frac{27}{4}+27\tan^2\left(\frac{3x}{2}\right)+\frac{81}{4}\tan^4\left(\frac{3x}{2}\right)$</td><td></td></tr>
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<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td rowspan="2">4(b)</td><td colspan="2">$$\left\{y\left(\frac{\pi}{4}\right)=\right\}1+3\left(\frac{\pi}{4}-\frac{\pi}{6}\right)+\frac{9}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)^2+9\left(\frac{\pi}{4}-\frac{\pi}{6}\right)^3$$ or $1+3\left(\frac{\pi}{12}\right)+\frac{9}{2}\left(\frac{\pi}{12}\right)^2+9\left(\frac{\pi}{12}\right)^3$
Substitutes $\frac{\pi}{4}$ into their expression for y of the correct form with at least the first three terms (series about $\frac{\pi}{6}$). Must have values (not unevaluated trig expressions).
If only a decimal value is given then it must be the correct awrt 2.26 to score M1 (2.255314325).
If there is no working they must obtain an expression with at least $a+b\pi+c\pi^2$ and correct exact ft a, b and c for their series or $1+\frac{\pi}{4}+c\pi^2$ with correct exact ft c</td><td>M1</td></tr>
<tr><td>$=1+\frac{\pi}{4}+\frac{\pi^2}{32}+\frac{\pi^3}{192}$ or $1+\frac{1}{4}\pi+\frac{1}{32}\pi^2+\frac{1}{192}\pi^3$</td><td>Correct answer or values for A (32) and B (192). Can be awarded if full marks were not scored in (a).</td><td>A1</td></tr>
<tr><td colspan="3"></td><td>(2)</td></tr>
<tr><td colspan="3"></td><td>Total 9</td></tr>
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Question Number	Scheme	Notes	Marks
5	$r^2=100\cos^2\theta+20\cos\theta\tan\theta+\tan^2\theta$	Any correct expression for r^2	**B1**
	$\left\{\frac{1}{2}\right\}\int_0^{\frac{\pi}{3}} r^2\,\mathrm{d}\theta=\left\{\frac{1}{2}\right\}\int_0^{\frac{\pi}{3}}\left(100\cos^2\theta+20\sin\theta+\tan^2\theta\right)\{\mathrm{d}\theta\}$	Attempts formula for the area with their r^2 which may not be expanded Condone missing $\frac{1}{2}$ and limits not required	**M1**
	$=\frac{1}{2}\int_0^{\frac{\pi}{3}}\left(50\left(1+\cos 2\theta\right)+20\sin\theta+\sec^2\theta-1\right)\{\mathrm{d}\theta\}$ **M1**: Uses $\cos^2\theta=\pm\frac{1}{2}\pm\frac{1}{2}\cos 2\theta$ **or** $\tan^2\theta=\pm\sec^2\theta\pm1$ in their r^2 **M1**: Uses both $\cos^2\theta=\pm\frac{1}{2}\pm\frac{1}{2}\cos 2\theta$ **and** $\tan^2\theta=\pm\sec^2\theta\pm1$ in their r^2 Both M marks can be scored without the integral and the $\frac{1}{2}$. Condone mixed variables. **A1**: Correct integral following $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$ and $\tan^2\theta=\sec^2\theta-1$. The $\cos\theta\tan\theta$ must be written as $\sin\theta$ (implied if appropriately integrated later). The $\frac{1}{2}$ is required (it may be seen later) but limits/ $\mathrm{d}\theta$ are not needed. Allow mixed variables if subsequent work recovers this.		**M1** **M1** **A1**
	$=\frac{1}{2}\left[49\theta+25\sin 2\theta-20\cos\theta+\tan\theta\right]_0^{\frac{\pi}{3}}$ or $\left[\frac{49}{2}\theta+\frac{25}{2}\sin 2\theta-10\cos\theta+\frac{1}{2}\tan\theta\right]_0^{\frac{\pi}{3}}$ **M1**: Achieves **three** of the following four integrated forms: $k\to k\theta$ (at least once), $\cos 2\theta\to\ldots\sin 2\theta$, $\sin\theta\to\ldots\cos\theta$, $\sec^2\theta\to\ldots\tan\theta$. Ignore other terms if 3 of the above are satisfied. No $\frac{1}{2}$ or limits required. Condone mixed variables. **A1**: Correct integration including the $\frac{1}{2}$ (may be seen later). Limits not required. May be unsimplified e.g., 49θ seen as $50\theta-\theta$. Allow mixed variables if subsequent work recovers this.		**M1** **A1**
	$=\frac{1}{2}\left(\frac{49\pi}{3}+25\sin\frac{2\pi}{3}-20\cos\frac{\pi}{3}+\tan\frac{\pi}{3}-\left(0+0-20+0\right)\right)$ $\left\{=\frac{1}{2}\left(\frac{49\pi}{3}+\frac{25\sqrt{3}}{2}-10+\sqrt{3}+20\right)\text{ or }\frac{49\pi}{6}+\frac{25\sqrt{3}}{4}-5+\frac{\sqrt{3}}{2}+10\right\}$ Applies the correct limits to an expression of the form $p\theta+q\sin 2\theta+r\cos\theta+s\tan\theta$ $\left(p,q,r,s\neq0\right)$ Allow slips but there must be a clear attempt to substitute, and they must only subtract the value of their *r*, e.g. if *r* = –20 work must have or imply ...– (–20) or +20. Allow mixed variables if the substitution recovers this.		**M1**
	$=\frac{1}{12}\left(98\pi+81\sqrt{3}+60\right)$	Correct answer or values for *a, b & c*	**A1**
	Note that there are other viable routes through the integration e.g., use of integration by parts		**(9)**
			Total 9

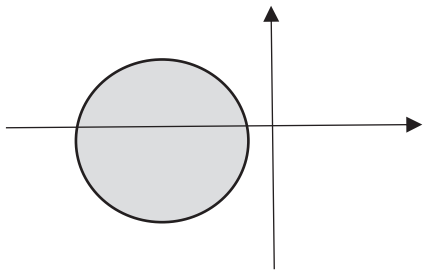
Question Number	Scheme	Notes	Marks
6	$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+13x=8e^{-3t} \qquad t\geqslant 0$		
(a)	$m^2+6m+13=0\Rightarrow m=\frac{-6\pm\sqrt{36-52}}{2}$ $\{=-3\pm 2i\}$	Forms correct auxiliary equation and obtains a correct numerical expression for at least one root by formula or uses CTS (apply usual CTS rule below). One correct root if no working	**M1**
	CTS rule: $m^2+6m+13=0\Rightarrow\left(m\pm\frac{6}{2}\right)^2\pm q\pm 13=0,\ q\neq 0\Rightarrow m=...$		
	CF examples: $(x=)\ e^{-3t}\left(A\cos 2t+B\sin 2t\right)$ or $(x=)\ Ae^{-3t}\cos(-2t)+Be^{-3t}\sin(-2t)$ or $(x=)\ Pe^{(-3+2i)t}+Qe^{(-3-2i)t}$ or $(x=)\ e^{-3t}\left(Pe^{2it}+Qe^{-2it}\right)$	Correct complementary function in any form, allow if the "$x=$" is missing or wrong and accept for this mark if the CF is given fully in terms of x instead of t.	**A1**
	PI: $\{x=\}\ \lambda e^{-3t}$	Correct form for the particular integral selected. Must include λe^{-3t} but accept with any extra terms that correctly disappear when coefficients found. Accept "PI=". If λe^{pt} is used $p=-3$ must be seen later.	**B1**
	$\frac{dx}{dt}=-3\lambda e^{-3t}$; $\frac{d^2x}{dt^2}=9\lambda e^{-3t}$ $\Rightarrow 9\lambda e^{-3t}+6\left(-3\lambda e^{-3t}\right)+13\lambda e^{-3t}=8e^{-3t}$	Differentiates a PI of **any** form twice (**provided** it has at least one constant and is a function of t) **and** substitutes into the equation. Allow only sign/coefficient errors only in the differentiation. Their PI must lead to non-zero derivatives.	**M1**
	$\Rightarrow 9\lambda-18\lambda+13\lambda=8\Rightarrow\lambda=...\ (2)$	Proceeds to find the value of the constant **following use of a PI of the correct form**. Any unnecessary extra terms in the PI must be found to be zero	**dM1**
	$x=\text{"}e^{-3t}\left(A\cos 2t+B\sin 2t\right)\text{"}+2e^{-3t}$	Correct general solution ft on **their CF only** – any CF **provided** it has at least one constant and is in terms of t. **Must have $x=\ldots$** Do not allow if their CF is miscopied or mathematically changed	**A1ft**
	Work with a PI of the form λte^{-3t} is B0M1dM0A0 max even if $2e^{-3t}$ is obtained. Only condone incorrect variables if they are recovered but refer to the note for the first A1.		**(6)**

<table>
<tr><th>Question Number</th><th colspan="2">Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td>6(b)</td><td colspan="2">$x=\frac{1}{2}$ at $t=0$
$\Rightarrow \frac{1}{2}=A+2 \quad \left(\Rightarrow A=-\frac{3}{2}\right)$</td><td>Uses the initial condition for x in their GS to find a linear equation in one or two constants. Allow for GS = CF or CF + PI and the constant may come from the +PI</td><td>M1</td></tr>
<tr><td></td><td colspan="3">$x=e^{-3t}\left(A\cos 2t+B\sin 2t\right)+2e^{-3t}$
$\frac{dx}{dt}=e^{-3t}\left(-2A\sin 2t+2B\cos 2t\right)-3e^{-3t}\left(A\cos 2t+B\sin 2t\right)-6e^{-3t}$
Uses the product rule to differentiate their real GS obtaining an expression in terms of t of the correct form for their GS (sign and coefficient errors only – so do not allow e.g., $...e^{pt}\rightarrow ...e^{qt}$). Allow for GS = CF or CF + PI and does not have to include constants.
If they work with a complex function e.g., $x=Pe^{(-3+2i)t}+Qe^{(-3-2i)t}+2e^{-3t}$ progress is unlikely.
This mark is not scored for work in (c)</td><td>M1</td></tr>
<tr><td></td><td colspan="3">$t=0,\frac{dx}{dt}=\frac{1}{2}\Rightarrow \frac{1}{2}=2B-3A-6\Rightarrow B=...(=1)$
Uses both initial conditions to find values for the 2 constants (no others) in their GS = (CF with 2 constants) + PI(no constants). One constant must be found to be non-zero.
Requires both previous M marks.</td><td>ddM1</td></tr>
<tr><td></td><td colspan="2">Examples:
$x=e^{-3t}\left(-\frac{3}{2}\cos 2t+\sin 2t\right)+2e^{-3t}$
or $x=e^{-3t}\left(-\frac{3}{2}\cos 2t+\sin 2t+2\right)$
or $x=2e^{-3t}-\frac{3}{2}e^{-3t}\cos 2t+e^{-3t}\sin 2t$</td><td>Correct particular solution in any form in terms of t.
Must be x = ... <u>unless</u> this was the only reason for final A0 in part (a) due to omission or e.g, "$y=$ ..." was used</td><td>A1</td></tr>
<tr><td></td><td colspan="3"></td><td>(4)</td></tr>
<tr><td>(c)</td><td colspan="3">$\frac{dx}{dt}=e^{-3t}\left(3\sin 2t+2\cos 2t\right)-3e^{-3t}\left(-\frac{3}{2}\cos 2t+\sin 2t\right)-6e^{-3t}=0$
Sets an expression for $\frac{dx}{dt}=0$. Accept with any unfound constants provided $\frac{dx}{dt}=f(t)$</td><td>M1</td></tr>
<tr><td></td><td colspan="3">$\left(3\sin 2t+2\cos 2t\right)-3\left(-\frac{3}{2}\cos 2t+\sin 2t\right)-6=0$
Achieves an equation of the form $a\sin bt+c\cos bt+d=0$ <u>or equivalent with terms uncollected</u>. One of a and c non-zero and b and d non-zero.
Must follow a GS = CF + PI where two constants were found for the CF and one for the PI. Requires previous M mark.</td><td>dM1</td></tr>
<tr><td></td><td colspan="3">$\cos 2t=\frac{12}{13}\Rightarrow t=0.1973955598...\Rightarrow x$ or $a=\frac{1}{2}e^{-3(0.1973...)}\left(4-3\times\frac{12}{13}+2\sin(2\times 0.1973...)\right)=..$
Finds a value of t from $\cos kt=c$ $\left(k\neq 1,\ -1<c<1\right)$ and uses their positive (or made positive) value of t to find a value of x (or a) via their PS. Accept a pair of stated values.
Requires both previous M marks.</td><td>ddM1</td></tr>
<tr><td></td><td colspan="2">x or $a=0.553(1164729...)$</td><td>awrt 0.553</td><td>A1</td></tr>
<tr><td></td><td colspan="3"></td><td>(4)</td></tr>
<tr><td></td><td colspan="3"></td><td>Total 14</td></tr>
</table>

<table>
<tr><th>Question Number</th><th>Scheme</th><th>Notes</th><th>Marks</th></tr>
<tr><td rowspan="7">**7(a)**

Way 1</td><td>$w=\dfrac{z-3}{2\mathrm{i}-z}\Rightarrow 2\mathrm{i}w-wz=z-3\Rightarrow z=...$</td><td>Attempts to make z the subject and obtains any f(w)</td><td>**M1**</td></tr>
<tr><td>$z=\dfrac{3+2\mathrm{i}w}{w+1}$ or $\dfrac{-3-2\mathrm{i}w}{-w-1}$</td><td>Any correct expression for z in terms of w</td><td>**A1**</td></tr>
<tr><td colspan="2">$=\dfrac{3+2\mathrm{i}u-2v}{u+\mathrm{i}v+1}\times\dfrac{u+1-\mathrm{i}v}{u+1-\mathrm{i}v}$
Applies $w = u + \mathrm{i}v$ and a correct multiplier for their z seen or implied by a correct result from their z. Denominator must have had a "w". Note alternative route below.</td><td>**M1**</td></tr>
<tr><td colspan="2">$x+\mathrm{i}y=\dfrac{3+2\mathrm{i}u-2v}{u+\mathrm{i}v+1}\times\dfrac{u+1-\mathrm{i}v}{u+1-\mathrm{i}v}=\dfrac{(3-2v)(u+1)+2uv+2u(u+1)\mathrm{i}-(3-2v)v\mathrm{i}}{(u+1)^2+v^2}$
$y=x+3$ oe $\Rightarrow\dfrac{2u(u+1)-(3-2v)v}{(u+1)^2+v^2}=\dfrac{(3-2v)(u+1)+2uv}{(u+1)^2+v^2}+3$
Multiplies, extracts real and imaginary parts and uses them in the equation $y = x + 3$ (oe) to produce an equation in u and v only – no "i"s. Condone $y = ...\mathrm{i}$ if recovered. Can follow slips with multiplier but denominator of z must have had a "w"
Note: Just $2u(u+1)-(3-2v)v=(3-2v)(u+1)+2uv+3$ is M0 (lost denominators)</td><td>**M1**</td></tr>
<tr><td>$2u(u+1)-(3-2v)v=(3-2v)(u+1)+2uv+3(u+1)^2+3v^2$
$\Rightarrow u^2+7u+v^2+v+6=0$</td><td>Expands and simplifies to obtain an equation of a circle with 4 or 5 real unlike terms.
All previous Ms required.</td><td>**dddM1**</td></tr>
<tr><td colspan="2">**Alternative for the above 3 marks** (note this could be done by equating expressions for y)
$x+\mathrm{i}y=\dfrac{3+2\mathrm{i}u-2v}{u+\mathrm{i}v+1}\Rightarrow\left(x+\mathrm{i}\left(x+3\right)\right)\left(u+1+\mathrm{i}v\right)=3+2u\mathrm{i}-2v$
M1: Applies $z=x+\mathrm{i}y$, uses $y = x + 3$ and cross multiplies
$x\left(u+1\right)-v\left(x+3\right)+\left(x+3\right)\left(u+1\right)\mathrm{i}+xv\mathrm{i}=3-2v+2u\mathrm{i}$
$\Rightarrow ux+x-vx-3v=3-2v,\quad ux+x+3u+3+xv=2u$
$\Rightarrow x=\dfrac{3+v}{u+1-v},\quad x=\dfrac{-u-3}{u+1+v}$
M1: Equates real and imaginary parts and makes x the subject twice
$\left(3+v\right)\left(u+1+v\right)=-\left(u+3\right)\left(u+1-v\right)\Rightarrow 3u+3+3v+uv+v+v^2=-u^2-u+uv-3u-3+3v$
$\Rightarrow u^2+v^2+7u+v+6=0$
M1: Equates expressions for x to obtain a circle equation with 4 or 5 real unlike terms</td><td></td></tr>
<tr><td colspan="2">$\Rightarrow\left(u+\dfrac{7}{2}\right)^2+\left(v+\dfrac{1}{2}\right)^2=\dfrac{49}{4}+\dfrac{1}{4}-6=\dfrac{13}{2}\Rightarrow$ centre: $\left(-\dfrac{7}{2},-\dfrac{1}{2}\right)$ radius: $\dfrac{\sqrt{26}}{2}$ or $\sqrt{\dfrac{13}{2}}$
M1: Extracts the centre **and/or** radius from their **circle** equation, however obtained, with 4 or 5 real unlike terms. Circle equation must not be in terms of z or w. They must get one correct coordinate (but condone wrong sign) or the correct radius for their circle.
May use $u^2+v^2+2gu+2fv+c=0\Rightarrow$ centre: $\left(-g,-f\right)$, radius $=\sqrt{g^2+f^2-c}$
A1: For a correct centre or radius **from a correct circle equation**
A1: For correct centre and radius **from a correct circle equation**
Centre as coordinates, x/u=..., y/v =... or as $-\dfrac{7}{2}-\dfrac{1}{2}\mathrm{i}$ and allow $\left(-\frac{7}{2},-\frac{1}{2}\mathrm{i}\right)$
Allow exact equivalents for coordinates/radius</td><td>**M1**
A1
A1</td></tr>
<tr><td></td><td colspan="2"></td><td>**(8)**</td></tr>
</table>

Question Number	Scheme	Notes	Marks
7(a) **Way 2**	$w=\dfrac{z-3}{2\text{i}-z}=\dfrac{x+\text{i}y-3}{2i-x-\text{i}y}=\dfrac{x-3+\text{i}(x+3)}{2\text{i}-x-\text{i}(x+3)}$ [Note that it is possible to replace *x* with *y* – 3]	M1: Uses $z = x + \text{i}y$ and $y = x + 3$ in the given transformation A1: Correct expression for w in terms of x	**M1** **A1**
	$\dfrac{x-3+\text{i}(x+3)}{-x-\text{i}(x+1)}=u+\text{i}v\Rightarrow x-3+\text{i}(x+3)=-xu+v(x+1)-\text{i}u(x+1)-\text{i}vx$	Applies $w = u + \text{i}v$ and multiplies	**M1**
	$x-3=-ux+vx+v,\quad x+3=-ux-u-vx$ $x=\dfrac{3+v}{1+u-v},\quad x=\dfrac{-3-u}{1+u+v}$	Equates real and imaginary parts and makes x the subject twice	**M1**
	$3+3u+3v+v+uv+v^2=-3-3u+3v-u-u^2+uv$ $\Rightarrow u^2+v^2+7u+v+6=0$	Equates expressions for x to obtain a circle equation with 4 or 5 real unlike terms. **All previous Ms required.**	**dddM1**
	$\Rightarrow\left(u+\dfrac{7}{2}\right)^2+\left(v+\dfrac{1}{2}\right)^2=\dfrac{49}{4}+\dfrac{1}{4}-6=\dfrac{13}{2}\Rightarrow$ centre: $\left(-\dfrac{7}{2},-\dfrac{1}{2}\right)$ radius: $\dfrac{\sqrt{26}}{2}$ or $\sqrt{\dfrac{13}{2}}$ **M1**: Applies a correct process to extract the centre and/or radius from a circle equation, however obtained, with 4 or 5 real unlike terms. One correct coordinate (but condone wrong sign) or radius correct for their circle. May use $u^2+v^2+2gu+2fv+c=0\Rightarrow$ centre $:(-g,-f)$, radius $=\sqrt{g^2+f^2-c}$ **A1**: For correct centre or radius from a correct circle equation **A1**: For correct centre and radius from a correct circle equation Centre as coordinates, *x*/*u*=..., *y*/*v* =... or as $-\dfrac{7}{2}-\dfrac{1}{2}\text{i}$ and allow $\left(-\frac{7}{2},-\frac{1}{2}\text{i}\right)$ **(8)**		**M1** **A1** **A1**
Way 3	e.g., 3 points on line are (0,3), (1,4) and (2,5) or $z_1=3\text{i},\ z_2=1+4\text{i},\ z_3=2+5\text{i}$	Attempts three points/complex numbers on $y = x + 3$ with 2 correct	**M1**
	$w=\dfrac{z-3}{2\text{i}-z}\Rightarrow w_1=\dfrac{3\text{i}-3}{-\text{i}}\quad w_2=\dfrac{-2+4\text{i}}{-1-2\text{i}}\quad w_3=\dfrac{-1+5\text{i}}{-2-3\text{i}}$	Correct transformed complex numbers	**A1**
	$w_1=\dfrac{3\text{i}-3}{-\text{i}}\times\dfrac{\text{i}}{\text{i}}\quad w_2=\dfrac{-2+4\text{i}}{-1-2\text{i}}\times\dfrac{-1+2\text{i}}{-1+2\text{i}}\quad w_3=\dfrac{-1+5\text{i}}{-2-3\text{i}}\times\dfrac{-2+3\text{i}}{-2+3\text{i}}$ At least two correct multipliers to remove "i" from denominator seen or implied (one if (–1, 2) used). Requires 2 correct points/complex numbers on line		**M1**
	$w_1=-3-3\text{i}\quad w_2=-\dfrac{6}{5}-\dfrac{8}{5}i\quad w_3=-1-\text{i}$	Two correct complex numbers in $a + \text{i}b$ form or as points	**M1**
	e.g., $x^2+y^2+2gx+2fy+c=0\Rightarrow$ $6g+6f-c=18$ $\frac{12}{5}g+\frac{16}{5}f-c=0$ $2g+2f-c=0$	Uses a correct general equation of a circle to form three simultaneous equations. **All previous Ms required.**	**dddM1**
	$\Rightarrow g=\dfrac{7}{2},\ f=\dfrac{1}{2},\ c=6\Rightarrow$ centre $(-g,-f):\left(-\dfrac{7}{2},-\dfrac{1}{2}\right)$ radius $=\sqrt{g^2+f^2-c}=\dfrac{\sqrt{26}}{2}$ or $\sqrt{\dfrac{13}{2}}$ **M1**: Solves and obtains at least one correct coordinate (but condone wrong sign) or radius for their constants **A1**: Correct centre or radius from correct work **A1**: Correct centre and radius from correct work **(8)**		**M1** **A1** **A1**

Question Number	Scheme	Notes	Marks
7(b) (i) & (ii)		M1: **Any circle with the whole interior indicated**. Ignore any inconsistencies with their stated centre, value for radius (which may have been negative) or circle equation. If shaded, consider the shaded area but if not allow any credible indication such as an "*R*" inside the circle unless they have **clearly** indicated a segment. A1: **Correct circle drawn in the correct position with whole interior shaded**. Entirely in quadrants 2 & 3 and centre if marked in Q3 (if not marked then more than half of the circle in Q3). Condone if it appears that the area above the x-axis is greater than the area below provided the centre is indicated in Q3. **Must be shaded** but does not require a label. Circumference may be dotted/dashed line. Ignore incorrect labelling of centre/axes/intersections but **requires full marks in (a).**	**M1 (B1 on ePen)** **A1 (B1 on ePen)**
			(2)
			Total 10

Question Number	Scheme	Notes	Marks
8(a)	Allow "single fraction" to be implied by sum/difference of fractions with same denominator or a product of fractions. No further fractions in numerator/denominator.		

	$\cot 2x\left\{+\tan x\right\} = \dfrac{\cos 2x}{\sin 2x}\left\{+\dfrac{\sin x}{\cos x}\right\}$	Uses $\cot 2x = \dfrac{\cos 2x}{\sin 2x}$ or e.g., $\dfrac{\cos 2x}{2\sin x\cos x}$	**M1**
	$\dfrac{\cos 2x + 2\sin^2 x}{2\sin x\cos x} \Rightarrow$ e.g., $\dfrac{1-2\sin^2 x+2\sin^2 x}{2\sin x\cos x}$ or $\dfrac{\cos^2 x-\sin^2 x+2\sin^2 x}{2\sin x\cos x}$ $\dfrac{2\cos^2 x-1+2\sin^2 x}{\sin 2x}$ or $\dfrac{\cos 2x+1-\cos 2x}{\sin 2x}$ OR $\dfrac{\cos 2x+\tan x\sin 2x}{\sin 2x} \Rightarrow$ $\Rightarrow \dfrac{\cos 2x+\frac{\sin x}{\cos x}\times 2\sin x\cos x}{\sin 2x} \Rightarrow$ e.g., $\dfrac{1-2\sin^2 x+2\sin^2 x}{\sin 2x}$ OR $\dfrac{\cos 2x\cos x+\sin x\sin 2x}{\sin 2x\cos x} \Rightarrow$ $\dfrac{\cos x}{\sin 2x\cos x}$ or $\dfrac{\cos^3 x-\sin^2 x\cos x+2\sin^2 x\cos x}{\sin 2x\cos x}$	Uses sufficient correct identities e.g., $\cos 2x = 1-2\sin^2 x$ $\cos 2x = \cos^2 x-\sin^2 x$ $\cos 2x = 2\cos^2 x-1$ $2\sin^2 x = 1-\cos 2x$ $\cos 2x\cos x+\sin x\sin 2x = \cos(2x-x)$ to obtain a correct single fraction with numerator in terms of $\sin x$ and/or $\cos x$ or "$\cos 2x+1-\cos 2x$". A qualifying fraction must be seen before $\dfrac{1}{2\sin x\cos x}$ or $\dfrac{1}{\sin 2x}$ Condone poor notation.	**A1 (M1 on ePen)**
	$= \dfrac{1}{2\sin x\cos x}$ or $\dfrac{1}{\sin 2x} = \operatorname{cosec} 2x$ *	Fully correct proof with one of the two intermediate fractions seen. All notation correct – no mixed or missing arguments or e.g. $\sin x^2$ for this mark.	**A1***
			(3)
Alt	$\cot 2x\left\{+\tan x\right\} = \dfrac{1-\tan^2 x}{2\tan x}\left\{+\tan x\right\}$	Uses $\cot 2x = \dfrac{1-\tan^2 x}{2\tan x}$	**M1**
	$\dfrac{1-\tan^2 x+2\tan^2 x}{2\tan x} \Rightarrow$ e.g., $\dfrac{\tan^2 x+1}{2\tan x} \Rightarrow \dfrac{\left(\frac{\sin x}{\cos x}\right)^2+1}{2\frac{\sin x}{\cos x}} \Rightarrow \dfrac{\cos x\left(\sin^2 x+\cos^2 x\right)}{2\cos^2 x\sin x}$ or $\dfrac{\tan^2 x+1}{2\tan x}\left\{\times\dfrac{\cos x}{\cos x}\right\} \Rightarrow \dfrac{\sin^2 x+\cos^2 x}{2\sin x\cos x}$ or $\dfrac{\sec^2 x}{2\tan x}$ or $\dfrac{\cos x}{2\cos^2 x\sin x}$	Uses correct identities e.g., $\tan x = \dfrac{\sin x}{\cos x}$ oe to obtain a correct single fraction in $\sin x$ and $\cos x$ but allow $\dfrac{\sec^2 x}{2\tan x}$ following use of $\sec^2 x = 1+\tan^2 x$ A qualifying fraction must be seen before $\dfrac{1}{2\sin x\cos x}$ or $\dfrac{1}{\sin 2x}$ Condone poor notation.	**A1 (M1 on ePen)**
	$\dfrac{1}{2\sin x\cos x}$ or $\dfrac{1}{\sin 2x} = \operatorname{cosec} 2x$ *	Fully correct proof with one of the two intermediate fractions seen. All notation correct – no mixed or missing arguments or e.g. $\sin x^2$ for this mark.	**A1***
			(3)

Question Number	Scheme	Notes	Marks
8(b)	Examples:		**M1** **A1**

	$y^2 = w\sin 2x \Rightarrow 2y\frac{dy}{dx} = \frac{dw}{dx}\sin 2x + 2w\cos 2x$ or $y = w^{\frac{1}{2}}(\sin 2x)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}w^{\frac{1}{2}}(\sin 2x)^{-\frac{1}{2}}(2\cos 2x) + \frac{1}{2}w^{-\frac{1}{2}}\frac{dw}{dx}(\sin 2x)^{\frac{1}{2}}$ or $w = \frac{y^2}{\sin 2x} \Rightarrow \frac{dw}{dx} = \frac{2y\sin 2x\frac{dy}{dx} - y^2.2\cos 2x}{\sin^2 2x}$ or $w = y^2\text{cosec}2x \Rightarrow 2y\frac{dy}{dx}\text{cosec}2x - 2y^2\text{cosec}2x\cot 2x$ **M1**: Attempts the differentiation of the given substitution using the product/quotient and chain rules **and** obtains an **equation** in $\frac{dy}{dx}$ and $\frac{dw}{dx}$ of the correct form (sign/coefficient errors only and allow sign errors with quotient/product rule). This mark is not available for work in $\frac{dy}{dw}$ or $\frac{dw}{dy}$ unless appropriate work follows to achieve an equation in $\frac{dy}{dx}$ and $\frac{dw}{dx}$ of the correct form. **A1**: Correct differentiation	
	$y\frac{dy}{dx} + y^2\tan x = \sin x \rightarrow$ e.g., $\frac{1}{2}\left(\frac{dw}{dx}\sin 2x + 2w\cos 2x\right) + w\sin 2x\tan x = \sin x$ A recognisable attempt to eliminate y from the original equation to obtain an equation involving $\frac{dw}{dx}$, w and x only . Not dependent.	**M1**
	$\Rightarrow \frac{dw}{dx} + 2w\left(\cot 2x + \tan x\right) = \frac{2\sin x}{\sin 2x}$ $\Rightarrow \frac{dw}{dx} + 2w\text{cosec } 2x = \sec x$ * Fully correct work leading to the given equation with $2w\left(\cot 2x + \tan x\right)$ or e.g., $2w\cot 2x + 2w\tan x$ clearly replaced by $2w\text{cosec } 2x$ but allow cot $2x$ written as $\frac{1}{\tan 2x}$ or $\frac{\cos 2x}{\sin 2x}$ and/or tan x written as $\frac{\sin x}{\cos x}$ If the result in (a) is not clearly used there must be full equivalent work. Allow use of "csc $2x$"	**A1***
		(4)

Question Number	Scheme	Notes	Marks

8(c)	$\frac{\mathrm{d}w}{\mathrm{d}x}+2w\operatorname{cosec}2x=\sec x\Rightarrow \text{IF}=\mathrm{e}^{2\int \operatorname{cosec}2x\,\mathrm{d}x}=\tan x$ or $\mathrm{e}^{-\ln(\operatorname{cosec}2x+\cot 2x)}\Rightarrow\frac{1}{\operatorname{cosec}2x+\cot 2x}$ or $\frac{1}{\cot x}$ or $\tan x$	**M1**: $\mathrm{e}^{2\int \operatorname{cosec}2x(\mathrm{d}x)}$ condoning omission of one or both "2"s **A1**: $\tan x$ oe Allow $k\tan x$ e.g., $\mathrm{e}^{2c}\tan x$ Not just $\mathrm{e}^{\ln(\tan x)}$	**M1** **A1**
	$\Rightarrow w"\tan x"=\int"\tan x"\sec x\,\{\mathrm{d}x\}$	Correctly applies their integrating factor to the equation, i.e., $\Rightarrow \text{IF}\times w=\int \text{IF}\times\sec x\,\{\mathrm{d}x\}$ Allow equivalents for $\sec x$. Condone "y" used for "w" for this mark.	**M1**
	$\Rightarrow w\tan x=\sec x(+c)$	Correct equation oe with or without constant.	**A1**
	Using $\text{IF}=\frac{1}{\operatorname{cosec}2x+\cot 2x}\Rightarrow$ RHS of $\int\frac{\sec x}{\operatorname{cosec}2x+\cot 2x}\mathrm{d}x$ which is likely to need rewriting as $\int\tan x\sec x\,\mathrm{d}x$ Note that IBP on $\sec x\tan x$ by writing it as $\sec^2 x\sin x$ can lead to $\sin x\tan x+\cos x(+c)$ Use Review for any attempts at integration you are unsure about.		
	e.g., $y^2=w\sin 2x$ and $w\tan x=\sec x+c\Rightarrow\frac{y^2}{\sin 2x}\tan x=\sec x+c$ $\Rightarrow y^2=\ldots\left\{\frac{\sin 2x}{\tan x}(\sec x+c)\right\}$ Substitutes for w correctly and reaches $y^2=\ldots$ Their $y^2=\ldots$ must be consistent with **their** equation in w and x that immediately followed their integration. **This mark requires both previous M marks and an attempt at integration that includes a "+ c"** A further example is: $w=\operatorname{cosec}x+\frac{c}{\tan x}\Rightarrow y^2=\operatorname{cosec}x\sin 2x+\frac{c\sin 2x}{\tan x}$		**ddM1**
	$\left\{\text{e.g., } y^2=\frac{2\sin x\cos^2 x}{\sin x}\left(\frac{1}{\cos x}+c\right)\Rightarrow\right\}$ $y^2=2\cos x+A\cos^2 x$ Any correct $y^2=\ldots$ equation with RHS fully in terms of $\cos x$. E.g. accept $y^2=2\cos x+2c\cos^2 x$ $\quad y^2=\cos x(2+A\cos x)$ $\quad y^2=2\cos^2 x\left(\frac{1}{\cos x}+c\right)$ Ignore any inconsistencies with the constant e.g., $2c$ later written as c		**A1**
			(6)
			Total 13