CAIE 9709 Paper 3 - Pure Mathematics

Compiled by: Dr Yu

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1 Algebra

- The modulus function $|x|$ is defined by

$$|x| = \begin{cases} x, & \text{if} \quad x \geq 0 \\ -x, & \text{if} \quad x < 0 \end{cases}$$

- The division algorithm for polynomials

$$\begin{array}{ccccccc} P(x) & = & D(x) & \times & Q(x) & + & R(x) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{dividend} & = & \text{divisor} & \times & \text{quotient} & + & \text{remainder} \end{array}$$

- The **factor theorem** 因式定理
  If for a polynomial $P(x)$, $P(c) = 0$, then $x - c$ is a factor of $P(x)$.
  If for a polynomial $P(x)$, $P\left(\frac{b}{a}\right) = 0$, then $ax - b$ is a factor of $P(x)$.

- The **remainder theorem** 余数定理
  If for a polynomial $P(x)$ is divided by $x - c$, the remainder is $P(c)$.
  If for a polynomial $P(x)$ is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$.

2 Logarithm and Exponentials 对数和指数

- The rules of logarithms If $y = a^x$, then $x = \log_a(y)$.

$$\log_a(a) = 1, \quad \log_a(1) = 0, \quad \log_a(a^x) = x, \quad a^{\log_a(x)} = x.$$

  For $x > 0$ and $y > 0$,

  Product rule: $\log_a(xy) = \log_a(x) + \log_a(y)$

  Division rule: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

  Power rule: $\log_a(x)^m = m\log_a(x)$

  Special case of power rule:

$$\log_a\left(\frac{1}{x}\right) = -\log_a(x) \quad \text{and} \quad \log_a\left(\frac{1}{x}\right)^n = -n\log_a(x)$$

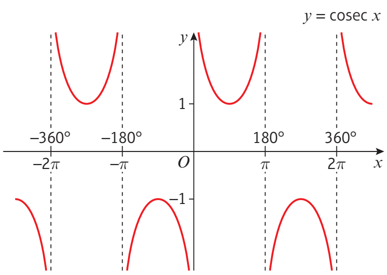
- Natural logarithms
  - ☐ Logarithms to the base of e are called natural logarithms.
  - ☐ $\mathrm{e} \approx 2.718$
  - ☐ $\ln(x)$ is used to represent $\log_{\mathrm{e}}(x)$.
  - ☐ If $y = \mathrm{e}^x$, then $x = \ln(y)$.
  - ☐ All the rules of logarithms apply for natural logarithms.

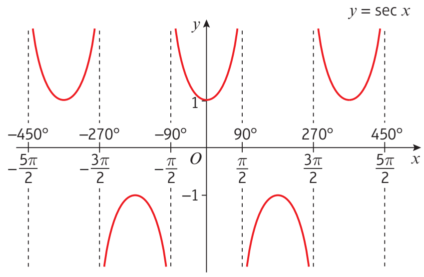
- Transforming a relationship to linear form
  - ☐ Logarithms can be used to convert relationships of the form $y = kx^n$ and $y = k(a^x)$ where k, a and n are constants, into straight lines of the form $Y = mX + c$, where X and Y are functions of x and y.

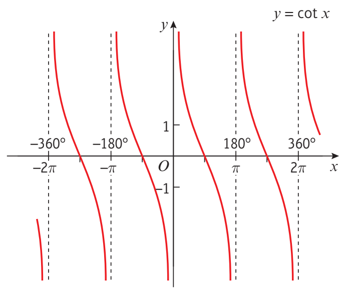
- 换元($\mathrm{e}^{2x} - 8\mathrm{e}^x + 15 = 0$)

3 Trigonometry 三角函数

- $\csc(x) = \frac{1}{\sin(x)}, \sec(x) = \frac{1}{\cos(x)}, \cot(x) = \frac{1}{\tan(x)}$







- Trigonometric identities
  - ☐ $1 + \tan^2(x) = \sec^2(x)$
  - ☐ $1 + \cot^2(x) = \csc^2(x)$
- Compound angle formula
  - ☐ $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
  - ☐ $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$
  - ☐ $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
  - ☐ $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
  - ☐ $\tan(A + B) = \dfrac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$
  - ☐ $\tan(A - B) = \dfrac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$
- Double angle formulae
  - ☐ $\sin(2A) = 2\sin(A)\cos(A)$
  - ☐ $\cos(2A) = \cos^2(A) - \sin^2(A) = 1 - 2\sin^2(A) = 2\cos^2(A) - 1$
  - ☐ $\tan(2A) = \dfrac{2\tan(A)}{1 - \tan^2(A)}$
- Expressing $a\sin(\theta) + b\cos(\theta)$ in the form $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$
  - ✓ $a\sin(\theta) \pm b\cos(\theta) = R\sin(\theta \pm \alpha)$
  - ✓ $a\cos(\theta) \pm b\sin(\theta) = R\cos(\theta \mp \alpha)$

  where $R = \sqrt{a^2 + b^2}$ and $\tan(\alpha) = \frac{b}{a}$.

4 Differentiation

- Product rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} = uv' + v'u$$

- Quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} = \frac{u'v - v'u}{v^2}$$

- Exponential functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x} = \mathrm{e}^{x} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{ax+b} = a\mathrm{e}^{ax+b} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{f}(x)} = \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}$$

- Logarithmic functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\ln(ax+b) = \frac{a}{ax+b} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\ln(\mathrm{f}(x)) = \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$$

- Trigonometric functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\sin(ax+b) = a\cos(ax+b)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\cos(ax+b) = -a\sin(ax+b)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\tan(ax+b) = a\sec^2(ax+b)$$

5 Integration

Integration formulae

☐ P1 basic integration

$$\int x^n \,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + \mathrm{c}$$

☐

$$\int \mathrm{e}^{x} \,\mathrm{d}x = \mathrm{e}^{x} + \mathrm{c} \quad \text{and} \quad \int \mathrm{e}^{ax+b} \,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b} + \mathrm{c}$$

☐

$$\int \frac{1}{x} \,\mathrm{d}x = \ln|x| + \mathrm{c} \quad \text{and} \quad \int \frac{1}{ax+b} \,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + \mathrm{c}$$

☐

$$\int \cos(x) \,\mathrm{d}x = \sin(x) + \mathrm{c} \quad \text{and} \quad \int \cos(ax+b) \,\mathrm{d}x = \frac{1}{a}\sin(ax+b) + \mathrm{c}$$

☐

$$\int \sin(x) \,\mathrm{d}x = -\cos(x) + \mathrm{c} \quad \text{and} \quad \int \sin(ax+b) \,\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + \mathrm{c}$$

☐

$$\int \sec^2(x) \,\mathrm{d}x = \tan(x) + \mathrm{c} \quad \text{and} \quad \int \sec^2(ax+b) \,\mathrm{d}x = \frac{1}{a}\tan(ax+b) + \mathrm{c}$$

6 Numerical solutions of equations 数值方法

- The process of finding a sequence of values that become closer and closer to a point of intersection of $y = x$ and $y = \mathrm{F}(x)$ is called an iterative process.
- Each time you generate a value of x you carry out an iteration.
- When the point of intersection is at $x = \alpha$ and your iterations are values that are getting closer and closer to α, you are converging to α.

- Start with x_1, work out $\mathrm{F}(x_1)$. Is this accurate enough? Yes $\rightarrow$ Stop, No $\rightarrow$
  Let $x_2 = \mathrm{F}(x_1)$, work out $\mathrm{F}(x_2)$. Is this accurate enough? Yes $\rightarrow$ Stop, No $\rightarrow$
  Let $x_3 = \mathrm{F}(x_2)$, work out $\mathrm{F}(x_3)$. Is this accurate enough? Yes $\rightarrow$ Stop, No $\rightarrow$
  Let $x_4 = \mathrm{F}(x_3)$, work out $\mathrm{F}(x_4)$. Is this accurate enough? Yes $\rightarrow$ Stop, No $\rightarrow$
  ...
  Let $x_{n+1} = \mathrm{F}(x_n)$, work out $\mathrm{F}(x_{n+1})$. Is this accurate enough? Yes $\rightarrow$ Stop.

- The relationship $x_{n+1} = \mathrm{F}(x_n)$ is called an **iterative formula**.

- The sequence of values of x given by this formula are called $x_1, x_2, x_3, \dots, x_n$.

- When the value found is accurate enough, then it is usually given a particular name such as α.

- The success and speed of the iterative process depends on the function chosen for $\mathrm{F}(x_n)$ and the value chosen for x_1.

7 Further Algebra

- Partial fractions

  Case 1: If the denominator of a proper algebraic fraction has two distinct linear factors, the fractions can be split into partial fractions using the rule

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

  Case 2: If the denominator or a proper algebraic fraction has a repeated linear factor, the fraction can be split into partial fractions using the rule

$$\frac{px+q}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

  Case 3: If the denominator of a proper algebraic fraction has a quadratic factor of the form cx^2+d that cannot be factorised, the fraction can be split into partial fractions using the rule

$$\frac{px+q}{(ax+b)(cx^2+d)} = \frac{A}{ax+b} + \frac{Bx+C}{cx^2+d}$$

  Case 4: If an algebraic fraction is improper we must first express it as the sum of a polynomial and a proper fraction and then split the proper fraction into partial fractions.

- Binomial expansion of $(1+x)^n$ for values of n that are **not positive integers**

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

  for $|x| < 1$.

- Binomial expansion of $(a+x)^n$ for values of n that are **not positive integers**

$$(a+x)^n = a^n\left[1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \dots\right] \qquad \text{for} \qquad \left|\frac{x}{a}\right| < 1$$

8 Further Calculus

- $$\int \frac{k\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = k\ln|\mathrm{f}(x)| + \mathrm{c}$$

- Integration by substitution 换元法
  Substitutions can sometimes be used to simplify the form of a function so that its integral can be easily recognised. When using a substitution, we must ensure that the integral is completely rewritten in terms of the new variable before integrating.

- Integrating rational functions
  Some rational functions can be split into partial fractions that can then be integrated.

- Integration by parts
  $$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

9 Vectors 空间向量

- For position vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

- The magnitude (length or size) is worked out using Pythagoras' theorem: When
  $$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \qquad |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

- Two vectors are parallel if one is a scalar multiple of the other.

- Problem solving with angles will involve the use of the **scalar product**. When
  $$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$
  $$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}|\cos(\theta)$$

- Vector equation of a line: $\mathbf{r} = \mathbf{a} + t\mathbf{b}$

- To find the point of intersection or two intersecting lines, solve the parametric equations of the lines simultaneously.

10 Differential equations 微分方程

- General procedure to solve a differential equation using the technique of separating the variables

  Step: 1 Separate the variables $\mathrm{f}(y)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{g}(x)$.

  Step: 2 Integrate both sides
  $$\int \mathrm{f}(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \mathrm{g}(x)\,\mathrm{d}x$$
  to get
  $$\int \mathrm{f}(y)\,\mathrm{d}y = \int \mathrm{g}(x)\,\mathrm{d}x$$

  Step: 3 Solve to find a general solution.

  Step: 4 Substitute any given conditions to find a particular solution.

  Step: 5 Check your answer by differentiating.

- Other points to remember
  - Take care with **modulus signs** in expressions such as $\ln|\mathrm{f}(x)|$.
  - When finding a general solution, check to see if any values excluded when the variables are separated are possible solutions.

11 Complex Numbers 复数

$$\mathrm{i}^2 = -1$$

Cartesian form: $x + \mathbf{i}y$ where x and y are real values.

- Arithmetic operations on $z_1 = a + b\mathbf{i}$ and $z_2 = c + d\mathbf{i}$:

  Addition: $z_1 + z_2 = (a + c) + (b + d)\mathbf{i}$

  Subtraction: $z_1 - z_2 = (a - c) + (b - d)\mathbf{i}$

  Multiplication: $z_1 z_2 = (ac - bd) + (ad + bc)\mathbf{i}$

  Division: $\frac{z_1}{z_2} = \frac{(ac+bd)+(bc-ad)\mathbf{i}}{c^2-d^2}$

  Modulus: $|z| = \sqrt{x^2 + y^2}$

  Conjugate:共轭 $z = a + b\mathbf{i}$ and $z^* = a - b\mathbf{i}$ are (complex) conjugates of each other.

  Argument: Found using a diagram with $\tan(\theta) = \frac{y}{x}$.

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2| \quad \text{and} \quad \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

- Polar forms
  - Modulus-argument form: $r(\cos(\theta) + \mathbf{i}\sin(\theta))$
  - Exponential form: $r\mathrm{e}^{\mathbf{i}\theta}$
- Roots of equations occur in complex conjugate pairs
  - Quadratic equations have 2 real or 2 complex roots of the form $x \pm \mathbf{i}y$ with $y \neq 0$.
  - Cubic equations have 3 real or 1 real and 2 complex roots of the form $x \pm \mathbf{i}y$ with $y \neq 0$.
  - Quartic equations have 4 real or 2 real and 2 complex roots of the form $x \pm \mathbf{i}y$ with $y \neq 0$, or 4 complex roots of the form $x \pm \mathbf{i}y$ with $y \neq 0$.
  - The cube roots of one are:

$$z = 1, \quad z = \frac{-1 + \mathbf{i}\sqrt{3}}{2}, \quad z = \frac{-1 - \mathbf{i}\sqrt{3}}{2}.$$