CAIE 9709 Paper 1 - Pure Mathematics

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1 Quadratics 二项式

2 Functions 函数

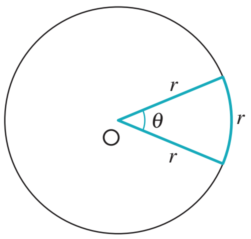
- Functions 函数
  - A function is a rule that maps (映射) each x value to just one y value for a defined set of input values.
  - A function can be either one-one or many-one.
  - The set of input values for a function is called the domain (定义域) of the function.
  - The set of output values for a function is called the range (值域) (or image set) of the function.
- Composite functions 复合函数
  - $\mathrm{fg}(x)$ means the function g acts on x first, then f acts on the result.
  - fg only exists if the range of g is contained within the domain of f.
  - In general, $\mathrm{fg}(x) \neq \mathrm{gf}(x)$.
- Inverse functions 反函数
  - The inverse of a function $\mathrm{f}(x)$ is the function that undoes what $\mathrm{f}(x)$ has done. $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$ or if $y = \mathrm{f}(x)$ then $x = \mathrm{f}^{-1}(y)$.
  - The inverse of the function $\mathrm{f}(x)$ is written as $\mathrm{f}^{-1}(x)$.
  - The steps for finding the inverse function are:
    - i: Write the function as $y =$.
    - ii: Interchange the x and y variables.
    - iii: Rearrange to make y the subject.
  - The domain of $\mathrm{f}^{-1}(x)$ is the range of $\mathrm{f}(x)$.
  - The range of $\mathrm{f}^{-1}(x)$ is the domain of $\mathrm{f}(x)$.
  - An inverse function $\mathrm{f}^{-1}(x)$ can exist if, and only if, the function $\mathrm{f}(x)$ is one-one.
  - The graphs of $\mathrm{f}(x)$ and $\mathrm{f}^{-1}(x)$ are reflections of each other in the line $y = x$.
  - If $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, then the function f is called a self-inverse function.
  - If f is self-inverse then $\mathrm{ff}(x) = x$.
  - The graph of a self-inverse function has $y = x$ as a line of symmetry.
- Transformations of functions
  - The graph of $y = \mathrm{f}(x) + a$ is a translation (平移) of $y = \mathrm{f}(x)$ by the vector $\binom{0}{a}$.
  - The graph of $y = \mathrm{f}(x + a)$ is a translation of $y = \mathrm{f}(x)$ by the vector $\binom{-a}{0}$.
  - The graph of $y = -\mathrm{f}(x)$ is a reflection (镜像) of the graph $y = \mathrm{f}(x)$ in the x-axis.
  - The graph of $y = \mathrm{f}(-x)$ is a reflection of the graph $y = \mathrm{f}(x)$ in the y-axis.
  - The graph of $y = a\mathrm{f}(x)$ is a stretch (拉伸) of $y = \mathrm{f}(x)$, stretch factor a, parallel to the y-axis.
  - The graph of $y = \mathrm{f}(ax)$ is a stretch of $y = \mathrm{f}(x)$, stretch factor $\frac{1}{a}$, parallel to the x-axis.
- Combining transformations
  - When two vertical transformations or two horizontal transformations are combined, the order in which they are applied may affect the outcome.

  - When one horizontal and one vertical transformation are combined, the order in which they are applied does not affect the outcome.
  - Vertical transformations follow the 'normal' order of operations, as used in arithmetic.
  - Horizontal transformations follow the *opposite* order to the 'normal' order of operations, as used in arithmetic.

3 Coordinate Geometry 平面几何

- For points $P(x_1, y_1)$ and $Q(x_2, y_2)$
  - Midpoint 中点 M is given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
  - Gradient 斜率 of PQ is $\frac{y_1-y_2}{x_1-x_2}$.
  - Length of segment 线段长 PQ is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.
- Parallel 平行 and perpendicular 垂直 lines
  - If the gradients of two parallel lines are m_1 and m_2, then $m_1 = m_2$.
  - If the gradients of two perpendicular lines are m_1 and m_2, then $m_1 \times m_2 = -1$.
- The equation of a straight line is:
$$y - y_1 = m(x - x_1),$$
  where m is the gradient and (x_1, y_1) is a point on the line.
- The equation of a circle is:
  - $(x-a)^2 + (y-b)^2 = r^2$, where (a, b) is the centre and r is the radius.
  - $x^2 + y^2 + 2gx + 2fy + c = 0$, where $(-g, -f)$ is the centre and $\sqrt{g^2+f^2-c}$ is the radius.

4 Circular Measure 角度/弧度



Radians 弧度 and degrees 角度

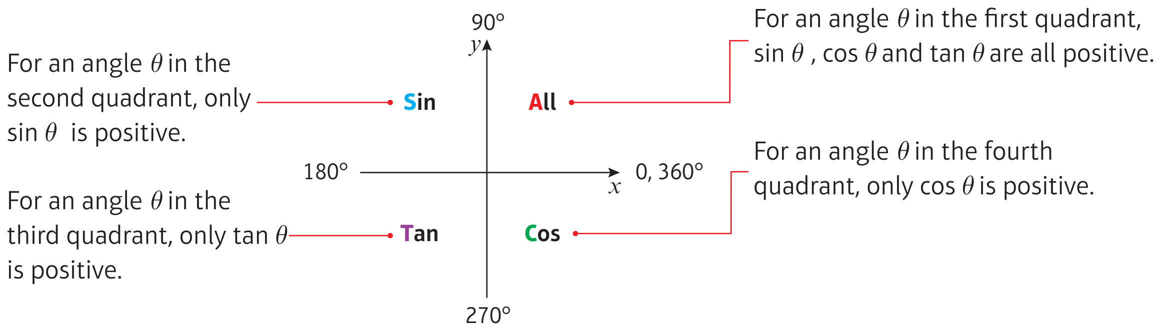
- One radian is the size of the angle subtended at the centre of a circle, radius r, by an arc of length r.
- π radians $= 180°$
- To change from degrees to radians, multiply by $\frac{\pi}{180}$.
- To change from radians to degrees, multiply by $\frac{180}{\pi}$.

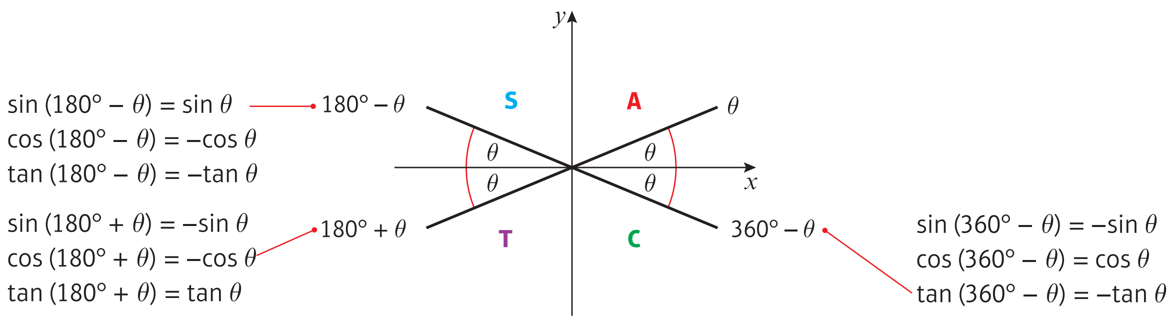
Arc length 弧长 and area of a sector 扇形

- When θ is measured in radians, the length of arc AB is $r\theta$.
- When θ is measured in radians, the area of sector AOB is $\frac{1}{2}r^2\theta$.

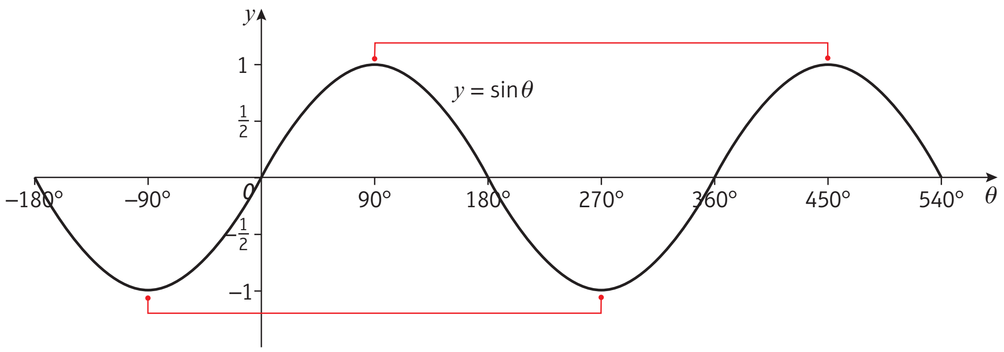
5 Trigonometry 三角函数

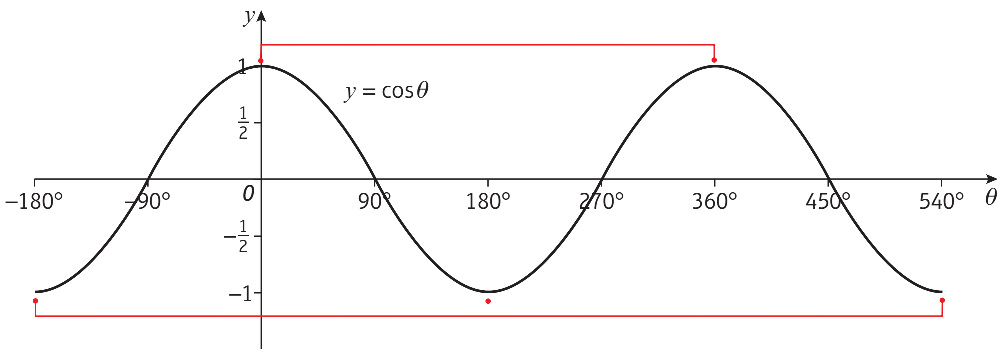
- Positive and negative angles
  - Angles measured **anticlockwise** 逆时针 from the positive x-direction are **positive**.
  - Angles measured **clockwise** 顺时针 from the positive x-direction are **negative**.
- Diagram showing where sin, cos and tan are positive. CAST 图 **A**ll **S**tudents **T**rust **C**ambridge

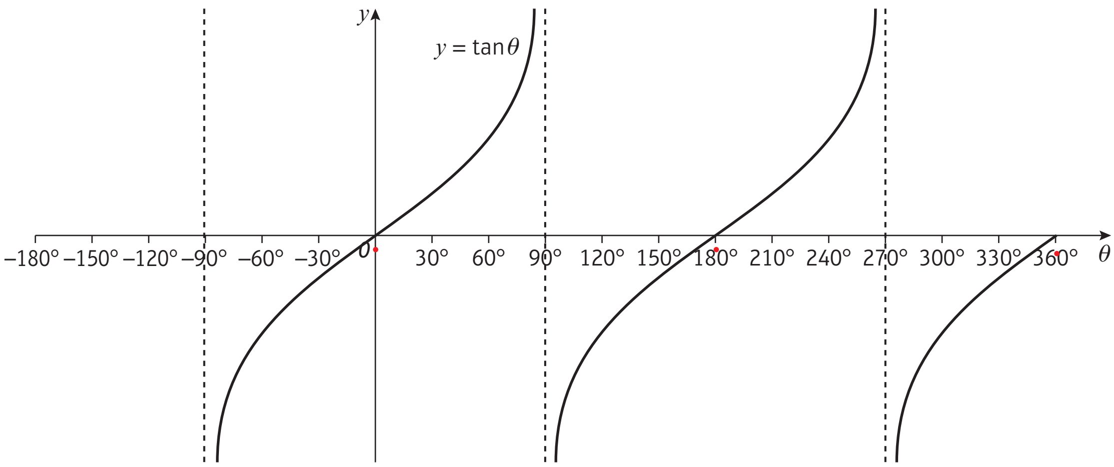




- Graphs of trigonometric functions 三角函数图 — 最好记住

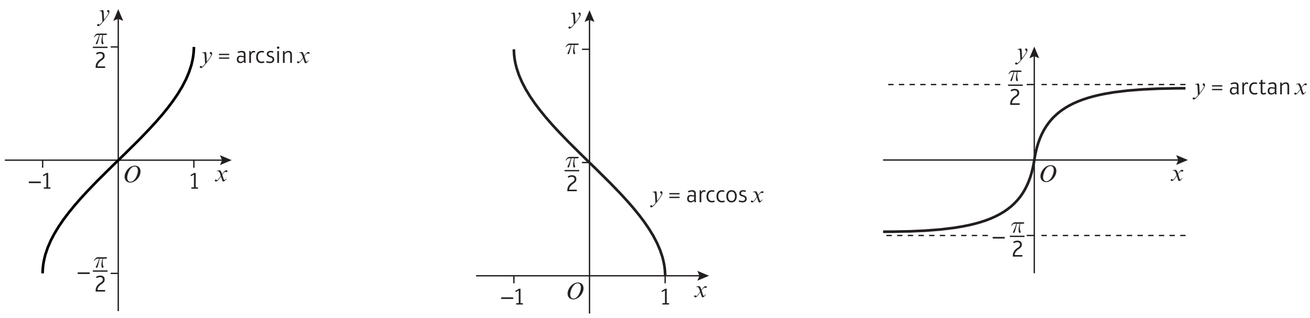






- The graph of $y = a\sin(x)$ is a stretch of $y = \sin(x)$, stretch factor a, parallel to the y-axis.
- The graph of $y = \sin(ax)$ is a stretch of $y = \sin(x)$, stretch factor $\frac{1}{a}$, parallel to the x-axis.
- The graph of $y = a + \sin(x)$ is a translation of $y = \sin(x)$ by the vector $\binom{0}{a}$.
- The graph of $y = \sin(x + a)$ is a translation of $y = \sin(x)$ by the vector $\binom{-a}{0}$.

• Inverse trigonometric functions 反三角函数 (三角函数的反函数)



• Trigonometric identities

- $\tan(x) = \frac{\sin(x)}{\cos(x)}$
- $\sin^2(x) + \cos^2(x) = 1$

6 Series 求和

• Binomial expansions 二次项展开
Binomial coefficients, denoted by ${}^nC_r$ or $\binom{n}{r}$, can be found using:

- Pascal's triangle 杨辉三角
- the formulae

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n \times (n-1) \times (n-2) \cdots \times (n-r+1)}{r \times (r-1) \times (r-2) \cdots \times 2 \times 1}$$

If n is a **positive integer**, the Binomial theorem states that:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$$

where the $(r+1)$th term $=\binom{n}{r}x^r$.
We can extend this rule to give:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n,$$

where the $(r+1)$th term $= \binom{n}{r}a^{n-r}b^r$.
We can also write the expansion of $(1+x)^n$ as:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

- Arithmetic series 等差数列
  For an arithmetic progression with first term a, common difference 公差 d and n terms:
  - the kth term is $a + (k-1)d$
  - the last term is $l = a + (n-1)d$
  - the sum of the first n terms is $S_n = \frac{n}{2}(a+l) = \frac{n}{2}(2a+(n-1)d)$.
- Geometric series 等比数列 (几何数列)
  For a geometric progression with first term a, common ratio 公比 r and n terms:
  - the kth term is ar^{k-1}
  - the last term is ar^{n-1}
  - sum of the terms is $S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$, if $|r|<1$.
  - When an infinite geometric series converges 收敛, $S_\infty = \frac{a}{1-r}$.

7 Differentiation 微分

- Gradient of a curve: $\frac{\mathrm{d}y}{\mathrm{d}x}$ represents the gradient of the curve $y = \mathrm{f}(x)$.
- The four rules of differentiation
  - $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$
  - $\frac{\mathrm{d}}{\mathrm{d}x}k\mathrm{f}(x) = k\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x)$
  - $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{f}(x) \pm \mathrm{g}(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x) \pm \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{g}(x)$
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$
- Second derivatives 二阶导 $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

8 Further Differentiation 进阶微分

Increasing and decreasing functions

- $y = \mathrm{f}(x)$ is increasing for a given interval of x if $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ throughout the interval.
- $y = \mathrm{f}(x)$ is decreasing for a given interval of x if $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ throughout the interval.

Stationary points

- Stationary points (turning points) of a function $y = \mathrm{f}(x)$ occur when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

First derivative test for maximum and minimum points

- At a maximum point: $\frac{dy}{dx} = 0$, the gradient is **positive** to the left of the maximum and **negative** to the right.
- At a minimum point: $\frac{dy}{dx} = 0$, the gradient is **negative** to the left of the minimum and **positive** to the right.

Second derivative test for maximum and minimum points

- If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, then the point is a maximum point.
- If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, then the point is a minimum point.
- If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, then the nature of the stationary point can be found using the first derivative test.

Connected rates of change 速率问题

- When two variables, x and y, both vary with a third variable, t, the three variables can be connected using the chain rule:
$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
- You may also need to use the rule:
$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

9 Integration 积分

Integration as the reverse of differentiation

- $$\frac{d}{dx}F(x) = f(x) \quad \Longrightarrow \quad \int f(x)\,dx = F(x) + c.$$

Integration formulae

- $$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \qquad \text{where } c \text{ is a constant and } n \neq -1.$$
- $$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c \qquad (n \neq -1, a \neq 0)$$

Rules for indefinite integration 不定积分

- $$\int kf(x)\,dx = k\int f(x)\,dx, \qquad \text{where } k \text{ is a constant.}$$
- $$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

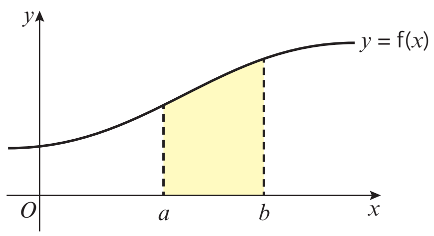
Rules for definite integration 定积分

- $$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

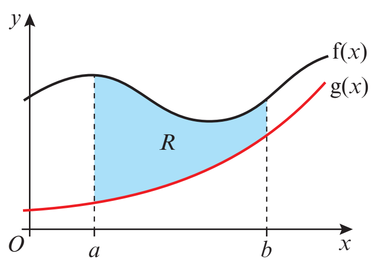
- $$\int_a^b k\mathrm{f}(x)\,\mathrm{d}x = k\int_a^b \mathrm{f}(x)\,\mathrm{d}x, \qquad \text{where } k \text{ is a constant.}$$
- $$\int_a^b [\mathrm{f}(x) \pm \mathrm{g}(x)]\,\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x \pm \int_a^b \mathrm{g}(x)\,\mathrm{d}x$$
- $$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = -\int_b^a \mathrm{f}(x)\,\mathrm{d}x$$

Area under a curve

- The area, A, bounded by the curve $y = \mathrm{f}(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by the formula:
$$A = \int_a^b \mathrm{f}(x)\,\mathrm{d}x \quad \text{when } \mathrm{f}(x) \geq 0$$



- The area, A, bounded by the curve $x = \mathrm{f}(y)$, the y-axis and the lines $y = a$ and $y = b$ is given by the formula:
$$A = \int_a^b \mathrm{f}(y)\,\mathrm{d}y \quad \text{when } \mathrm{f}(y) \geq 0$$
- The area, A, enclosed between $y = \mathrm{f}(x)$ and $y = \mathrm{g}(y)$ is given by the formula:
$$A = \int_a^b [\mathrm{f}(x) - \mathrm{g}(x)]\,\mathrm{d}x$$
where a and b are the x-coordinates of the points of intersection of the functions f and g.



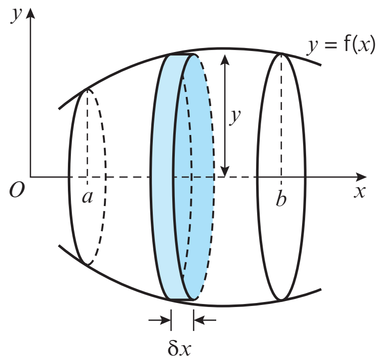
Improper integrals 无穷积分

- Integrals of the form $\int_a^\infty \mathrm{f}(x)\,\mathrm{d}x$ can be evaluated by replacing the infinite limit with a finite value, X, and then taking the limit as $X \to \infty$, provided the limit exists.
- Integrals of the form $\int_{-\infty}^b \mathrm{f}(x)\,\mathrm{d}x$ can be evaluated by replacing the infinite limit with a finite value, X, and then taking the limit as $X \to -\infty$, provided the limit exists.
- Integrals of the form $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ where $\mathrm{f}(x)$ is not defined when $x = a$ can be evaluated by replacing the limit a with an X and then taking the limit as $X \to a$, provided the limit exists.

- Integrals of the form $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ where $\mathrm{f}(x)$ is not defined when $x = b$ can be evaluated by replacing the limit b with an X and then taking the limit as $X \to b$, provided the limit exists.

Volume of revolution

- The volume, V, obtained when the function $y = \mathrm{f}(x)$ is rotated through 360° about the x-axis between the boundary values $x = a$ and $x = b$ is given by the formula $V = \int_a^b \pi y^2\,\mathrm{d}x$.



- The volume, V, obtained when the function $x = \mathrm{f}(y)$ is rotated through 360° about the y-axis between the boundary values $y = a$ and $y = b$ is given by the formula $V = \int_a^b \pi x^2\,\mathrm{d}y$.