1 It is given that $y = \sinh(x^2) + \cosh(x^2)$.

(a) Use standard results from the list of formulae (MF19) to find the Maclaurin's series for y in terms of x up to and including the term in x^4. [2]

....

....

....

....

....

....

....

....

(b) Deduce the value of $\frac{d^4y}{dx^4}$ when $x = 0$. [1]

....

....

....

(c) Use your answer to part **(a)** to find an approximation to $\int_0^{\frac{1}{2}} y\,dx$, giving your answer as a rational fraction in its lowest terms. [2]

....

....

....

....

....

....

....

....

....

....

....

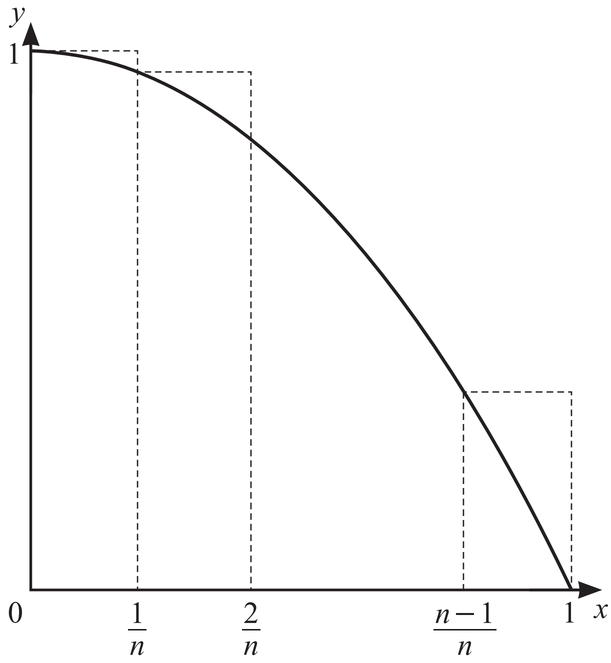
....

2 Find the solution of the differential equation

$$\frac{dy}{dx} + \frac{4x^3 y}{x^4 + 5} = 6x$$

for which $y = 1$ when $x = 1$. Give your answer in the form $y = f(x)$. [7]

3



The diagram shows the curve with equation $y = 1 - x^2$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of the rectangles, show that

$$\int_0^1 (1 - x^2)\,dx < \frac{4n^2 + 3n - 1}{6n^2}.$$ [4]

....

....

....

....

....

....

....

....

....

....

....

....

....

(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 (1-x^2)\,dx$. [4]

4 **(a)** Write down all the roots of the equation $x^5 - 1 = 0$. [2]

....

....

....

(b) Use de Moivre's theorem to show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. [4]

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

....

(c) Use the results of parts **(a)** and **(b)** to express each real root of the equation

$$8x^9 - 8x^7 + x^5 - 8x^4 + 8x^2 - 1 = 0$$

in the form $\cos k\pi$, where k is a rational number. [4]

5 The curve C has parametric equations

$$x = 3t + 2t^{-1} + at^3, \quad y = 4t - \tfrac{3}{2}t^{-1} + bt^3, \quad \text{for } 1 \leqslant t \leqslant 2,$$

where a and b are constants.

(a) It is given that $a = \frac{2}{3}$ and $b = -\frac{1}{2}$.

Show that $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{25}{4}(t^2 + t^{-2})^2$ and find the exact length of C. [6]

(b) It is given instead that $a = b = 0$.

Find the value of $\dfrac{d^2y}{dx^2}$ when $t = 1$. [4]

6 The matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 6 & 6 \\ 0 & 2 & 6 \\ 0 & 0 & -3 \end{pmatrix}.$$

(a) Use the characteristic equation of **P** to find $\mathbf{P}^{-1}$. [5]

(b) Find the matrix $\mathbf{A}$ such that

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$ [4]

(c) State the eigenvalues and corresponding eigenvectors of $\mathbf{A}^3$. [2]

7 It is given that $y = x^2 w$ and

$$x^2 \frac{d^2w}{dx^2} + 4x(x+1)\frac{dw}{dx} + (5x^2 + 8x + 2)w = 5x^2 + 4x + 2.$$

(a) Show that

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5x^2 + 4x + 2.$$ [4]

(b) Find the general solution for w in terms of x. [7]

8 **(a)** Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1 - \tanh^2 x = \operatorname{sech}^2 x.$$ [3]

(b) Using the substitution $u = \tanh x$, or otherwise, find $\int \operatorname{sech}^2 x \tanh^2 x \, dx$. [2]

It is given that, for $n \geqslant 0$, $I_n = \int_0^{\ln 3} \operatorname{sech}^n x \tanh^2 x \, dx$.

(c) Show that, for $n \geqslant 2$,

$$(n+1)I_n = \left(\tfrac{4}{5}\right)^3 \left(\tfrac{3}{5}\right)^{n-2} + (n-2)I_{n-2}.$$ [5]

[You may use the result that $\frac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$.]

(d) Find the value of I_4. [3]

