1   (a) By differentiating e^{-x^2}, find the Maclaurin's series for e^{-x^2} up to and including the term in x^2.

[5]

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(b) Deduce an approximation to $\int_0^{\frac{1}{5}} e^{-x^2}\,dx$, giving your answer as a rational fraction in its lowest terms.
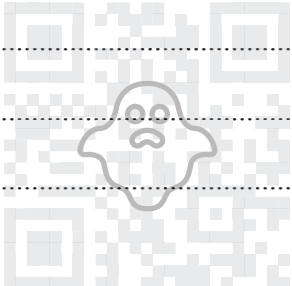
[2]

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2 The variables x and y are related by the differential equation

$$9\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = 3x^2 + 30x.$$

(a) Find the general solution for y in terms of x. [6]

(b) State an approximate solution for large positive values of x. [1]

3 **(a)** Show that the system of equations

$$x - 2y - 4z = 1,$$
$$x - 2y + kz = 1,$$
$$-x + 2y + 2z = 1,$$

where k is a constant, does not have a unique solution. [2]

(b) Given that $k = -4$, show that the system of equations in part **(a)** is consistent. Interpret this situation geometrically. [3]
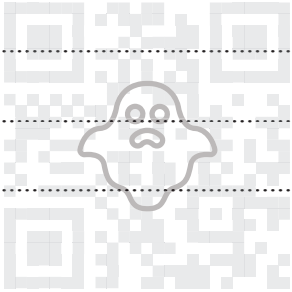
(c) Given instead that $k = -2$, show that the system of equations in part **(a)** is inconsistent. Interpret this situation geometrically. [2]

(d) For the case where $k \neq -2$ and $k \neq -4$, show that the system of equations in part **(a)** is inconsistent. Interpret this situation geometrically. [2]
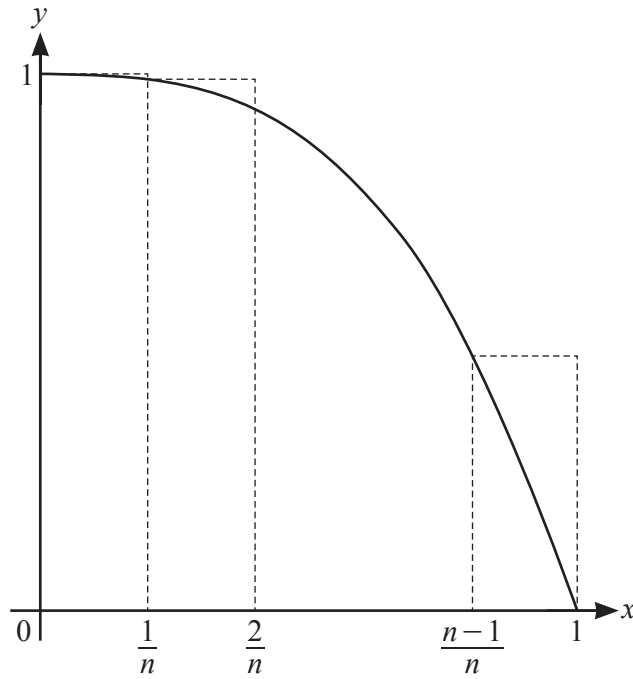
4

The diagram shows the curve with equation $y = 1 - x^3$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\dfrac{1}{n}$.

(a) By considering the sum of the areas of the rectangles, show that

$$\int_0^1 (1 - x^3)\,dx \leqslant \frac{3n^2 + 2n - 1}{4n^2}.$$ [4]

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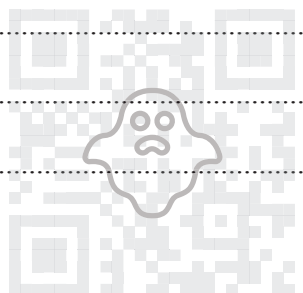
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(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 (1-x^3)\,\mathrm{d}x$. [4]

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5 It is given that

$$x = \sinh^{-1} t, \quad y = \cos^{-1} t,$$

where $-1 < t < 1$.

(a) By differentiating $\cos y$ with respect to t, show that $\dfrac{dy}{dt} = -\dfrac{1}{\sqrt{1-t^2}}$. [4]

(b) Find $\dfrac{d^2y}{dx^2}$ in terms of t, simplifying your answer. [5]

6 **(a)** Use de Moivre's theorem to show that $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$. [5]

(b) Find the solution of the differential equation

$$\frac{dy}{d\theta} + y\cot\theta = \sin^3\theta$$

for which $y = 0$ when $\theta = \frac{1}{2}\pi$. [6]

7 The matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) State the eigenvalues of **P**. [1]

(b) Use the characteristic equation of **P** to find $\mathbf{P}^{-1}$. [4]
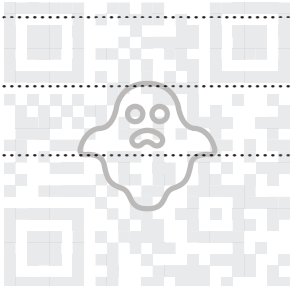
The 3×3 matrix $\mathbf{A}$ has distinct eigenvalues b, -1, 1 with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

respectively.

(c) Find $\mathbf{A}$ in terms of b. [4]

8   **(a)**   Sketch the graph of $y = \coth x$ for $x > 0$ and state the equations of the asymptotes.   [2]

(b)   Starting from the definitions of coth and cosech in terms of exponentials, prove that

$$\coth^2 x - \operatorname{cosech}^2 x = 1.$$ [3]

The curve C has equation $y = \ln\coth\left(\tfrac{1}{2}x\right)$ for $x > 0$.

(c) Show that $\dfrac{dy}{dx} = -\operatorname{cosech} x$.   [3]

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(d) It is given that the arc length of C from $x = a$ to $x = 2a$ is $\ln 4$, where a is a positive constant.

Show that $\cosh a = 2$ and find, in logarithmic form, the exact value of a.   [7]

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