1 **(a)** Find a and b such that

$$z^8 - iz^5 - z^3 + i = (z^5 - a)(z^3 - b).$$ [1]

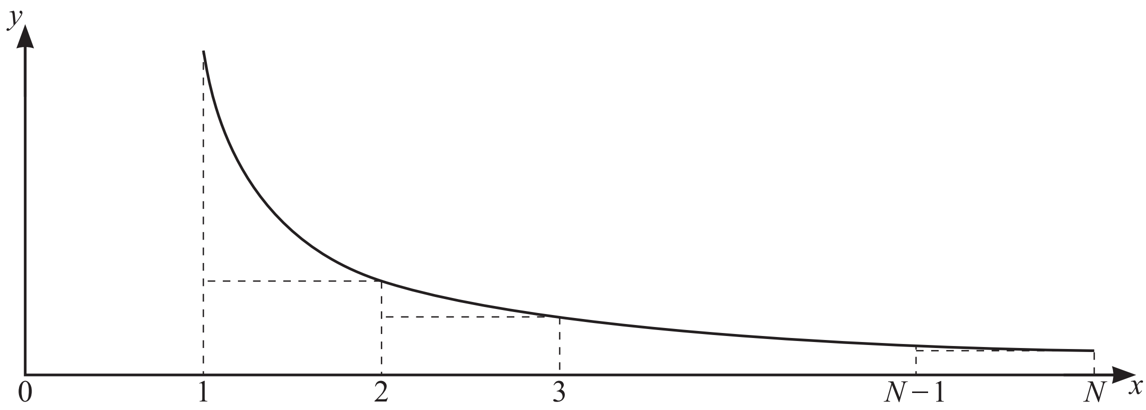
(b) Hence find the roots of

$$z^8 - iz^5 - z^3 + i = 0,$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [6]

2 Find the Maclaurin's series for $\ln\cosh x$ up to and including the term in x^4. [7]

3



The diagram shows the curve $y = \frac{x}{2x^2 - 1}$ for $x \geqslant 1$, together with a set of $N - 1$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{N} \frac{r}{2r^2 - 1} < \tfrac{1}{4}\ln(2N^2 - 1) + 1.$$ [7]

(b) Use a similar method to find, in terms of N, a lower bound for $\sum_{r=1}^{N} \frac{r}{2r^2 - 1}$. [3]

4 By considering the binomial expansions of $\left(z+\frac{1}{z}\right)^5$ and $\left(z-\frac{1}{z}\right)^5$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\tan^5\theta = \frac{\sin 5\theta - a\sin 3\theta + b\sin\theta}{\cos 5\theta + a\cos 3\theta + b\cos\theta},$$

where a and b are integers to be determined. [7]

5 The variables x and y are related by the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 4e^{-x}.$$

(a) Find the value of the constant k such that $y = kxe^{-x}$ is a particular integral of the differential equation. [4]

(b) Find the solution of the differential equation for which $y = \frac{dy}{dx} = \frac{1}{2}$ when $x = 0$. [6]

6 **(a)** Starting from the definitions of sinh and cosh in terms of exponentials, prove that

$$2\sinh^2 x = \cosh 2x - 1.$$ [3]

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(b) Find the solution to the differential equation

$$\frac{dy}{dx} + y\coth x = 4\sinh x$$

for which $y = 1$ when $x = \ln 3$. [7]

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7 The integral I_n, where n is an integer, is defined by $I_n = \int_0^{\frac{3}{2}} (4+x^2)^{-\frac{1}{2}n}\,dx$.

(a) Find the exact value of I_1, expressing your answer in logarithmic form. [3]

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(b) By considering $\frac{d}{dx}\left(x(4+x^2)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$4nI_{n+2} = \tfrac{3}{2}\left(\tfrac{2}{5}\right)^n + (n-1)I_n.$$ [5]

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(c) Find the value of I_5. [3]

8 **(a)** Find the value of a for which the system of equations

$$\begin{aligned} 13x + 18y - 28z &= 0, \\ -4x - ay + 8z &= 0, \\ 2x + 6y - 5z &= 0, \end{aligned}$$

does not have a unique solution. [2]

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The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix}.$$

(b) Find the eigenvalue of **A** corresponding to the eigenvector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. [1]

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(c) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [8]

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(d) Use the characteristic equation of **A** to find $\mathbf{A}^{-1}$ in terms of **A**. [2]