1 **(a)** Given that a is an integer, show that the system of equations

$$\begin{aligned} ax + 3y + z &= 14, \\ 2x + y + 3z &= 0, \\ -x + 2y - 5z &= 17, \end{aligned}$$

has a unique solution and interpret this situation geometrically. [4]

(b) Find the value of a for which $x = 1,\ y = 4,\ z = -2$ is the solution to the system of equations in part **(a)**. [1]

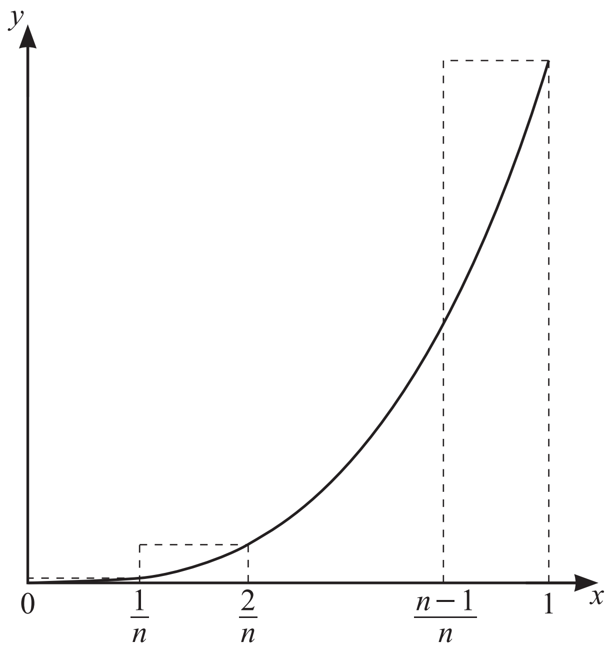
2 The variables x and y are related by the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2x + 1.$$

(a) Find the general solution for y in terms of x. [6]

(b) State an approximate solution for large positive values of x. [1]

3



The diagram shows the curve with equation $y = x^3$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 x^3\,dx < U_n$, where

$$U_n = \left(\frac{n+1}{2n}\right)^2.$$ [4]

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(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 x^3\,dx$. [4]

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(c) Find the least value of n such that $U_n - L_n < 10^{-3}$. [2]

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4 Find the solution of the differential equation

$$\sin\theta\frac{dy}{d\theta}+y=\tan\tfrac{1}{2}\theta,$$

where $0<\theta<\pi$, given that $y=1$ when $\theta=\frac{1}{2}\pi$. Give your answer in the form $y=f(\theta)$. [9]

[You may use without proof the result that $\int \operatorname{cosec}\theta\,d\theta=\ln\tan\frac{1}{2}\theta$.]

5 (a) State the sum of the series $z + z^2 + z^3 + ... + z^n$, for $z \neq 1$. [1]

(b) Given that z is an nth root of unity and $z \neq 1$, deduce that $1 + z + z^2 + ... + z^{n-1} = 0$. [2]

(c) Given instead that $z = \frac{1}{3}(\cos\theta + \mathrm{i}\sin\theta)$, use de Moivre's theorem to show that

$$\sum_{m=1}^{\infty} 3^{-m} \cos m\theta = \frac{3\cos\theta - 1}{10 - 6\cos\theta}.$$ [7]

6 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 5 & -\frac{22}{3} & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^2 = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of **A** to find $\mathbf{A}^3$. [4]

7 **(a)** It is given that $y = \operatorname{sech}^{-1}\left(x+\frac{1}{2}\right)$.

Express $\cosh y$ in terms of x and hence show that $\sinh y \frac{dy}{dx} = -\frac{1}{\left(x+\frac{1}{2}\right)^2}$. [3]

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(b) Find the first three terms in the Maclaurin's series for $\operatorname{sech}^{-1}\left(x+\frac{1}{2}\right)$ in the form

$$\ln a + bx + cx^2,$$

where a, b and c are constants to be determined. [7]

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8 The curve C has parametric equations

$$x = 2\cosh t, \quad y = \tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t, \quad \text{for } 0 \leqslant t \leqslant 1.$$

(a) Find $\dfrac{dx}{dt}$ and show that $\dfrac{dy}{dt} = 1 - \sinh^2 t$. [3]

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The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by A.

(b) (i) Show that $A = \pi\displaystyle\int_0^1 \left(\tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t\right)(1 + \cosh 2t)\,dt$. [4]

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(ii) Hence find A in terms of π, $\sinh 2$ and $\cosh 2$. [6]