1 Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} - 8\frac{dx}{dt} - 9x = 9e^{8t}.$$

[6]

2 Let $I_n = \int_0^1 (1+3x)^n e^{-3x}\,dx$, where n is an integer.

(a) Show that $3I_n = 1 - 4^n e^{-3} + 3nI_{n-1}$. [3]

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(b) Find the exact value of I_2. [3]

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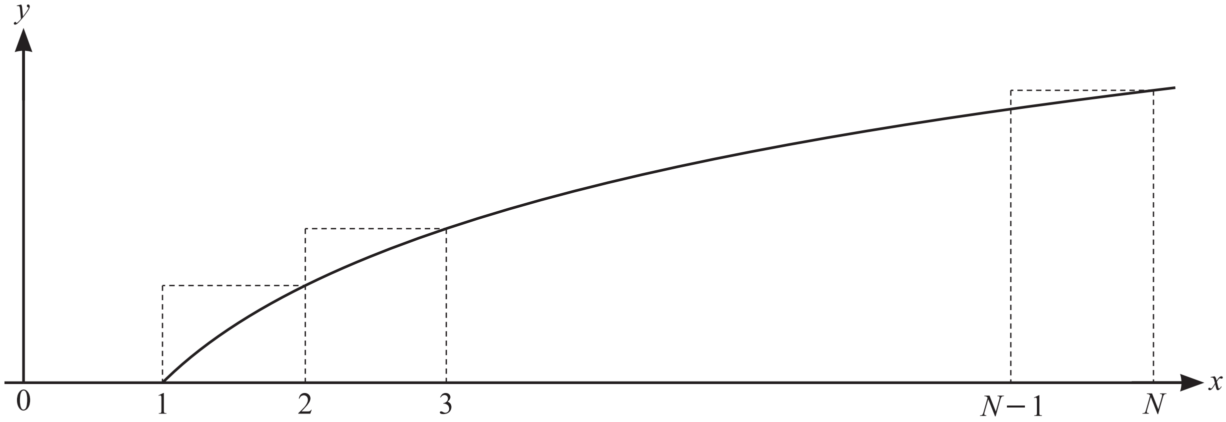
3 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 7 \\ 0 & 6 & 0 \\ 7 & 7 & 5 \end{pmatrix}.$$

(a) Find the eigenvalues of **A**. [4]

(b) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$. [4]

4



The diagram shows the curve with equation $y = \ln x$ for $x \geqslant 1$, together with a set of $(N-1)$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\ln N! > N \ln N - N + 1.$$ [5]

(b) Use a similar method to find, in terms of N, an upper bound for $\ln N!$. [3]

5 The curve C has parametric equations

$$x = \frac{1}{2}t^2 - \ln t, \qquad y = 2t + 1, \qquad \text{for } \frac{1}{2} \leqslant t \leqslant 2.$$

(a) Find the exact length of C. [5]

(b) Find $\frac{d^2y}{dx^2}$ in terms of t, simplifying your answer. [4]

6 **(a)** Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1 - \tanh^2\theta = \text{sech}^2\theta.$$ [3]

The variables x and y are such that $\tanh y = \cos\left(x + \frac{1}{4}\pi\right)$, for $-\frac{1}{4}\pi < x < \frac{3}{4}\pi$.

(b) By differentiating the equation $\tanh y = \cos\left(x + \frac{1}{4}\pi\right)$ with respect to x, show that

$$\frac{dy}{dx} = -\text{cosec}\left(x + \frac{1}{4}\pi\right).$$ [4]

(c) Hence find the first three terms in the Maclaurin's series for $\tanh^{-1}\left(\cos\left(x+\frac{1}{4}\pi\right)\right)$ in the form $\frac{1}{2}\ln a + bx + cx^2$, giving the exact values of the constants a, b and c. [5]

7 **(a)** Show that an appropriate integrating factor for

$$(x^2+1)\frac{dy}{dx}+y\sqrt{x^2+1}=x^2-x\sqrt{x^2+1}$$

is $x+\sqrt{x^2+1}$. [4]

(b) Hence find the solution of the differential equation

$$(x^2+1)\frac{dy}{dx}+y\sqrt{x^2+1}=x^2-x\sqrt{x^2+1}$$

for which $y = \ln 2$ when $x = 0$. Give your answer in the form $y = f(x)$. [7]

8 **(a)** Use de Moivre's theorem to show that $\sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$. [6]

It is given that $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$.

(b) Find the exact value of $\int_0^{\frac{1}{3}\pi}\left(\cos^6(\tfrac{1}{4}x) + \sin^6(\tfrac{1}{4}x)\right)\mathrm{d}x$. [4]

(c) Express each root of the equation $16c^6 + 16(1-c^2)^3 - 13 = 0$ in the form $\cos k\pi$, where k is a rational number. [5]