1 Find the solution of the differential equation

$$\frac{dy}{dx} + 5y = e^{-7x}$$

for which $y = 0$ when $x = 0$. Give your answer in the form $y = f(x)$. [6]

2 It is given that $y = 2^x$.

(a) By differentiating $\ln y$ with respect to x, show that $\frac{dy}{dx} = 2^x \ln 2$. [3]

(b) Write down $\frac{d^2y}{dx^2}$. [1]

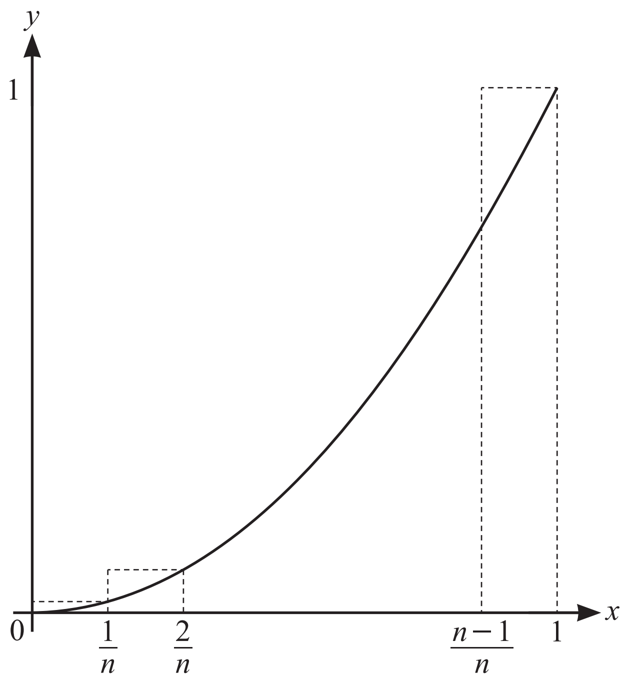
(c) Hence find the first three terms in the Maclaurin's series for 2^x. [3]

3 **(a)** Find the roots of the equation $z^3 = -1 - \mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [5]

Let $w = z_1^{3k} + z_2^{3k} + z_3^{3k}$, where k is a positive integer and z_1, z_2, z_3 are the roots of $z^3 = -1 - \mathrm{i}$.

(b) Express w in the form $R\mathrm{e}^{\mathrm{i}\alpha}$, where $R > 0$, giving R and α in terms of k. [3]

4



The diagram shows the curve with equation $y = x^2$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that

$$\int_0^1 x^2 \, dx < \frac{2n^2 + 3n + 1}{6n^2}.$$ [4]

(b) Use a similar method to find, in terms of n, a lower bound for $\int_0^1 x^2\,dx$. [4]

5 The curves $C_1 : y = \cosh x$ and $C_2 : y = \sinh 2x$ intersect at the point where $x = a$.

(a) Find the exact value of a, giving your answer in logarithmic form. [4]

(b) Sketch C_1 and C_2 on the same diagram. [2]

(c) Find the exact value of the length of the arc of C_1 from $x = 0$ to $x = a$. [5]

6 The integral I_n, where n is an integer, is defined by $I_n = \int_0^{\frac{1}{2}} \left(1 - x^2\right)^{-\frac{1}{2}n} \mathrm{d}x$.

(a) Find the exact value of I_1. [2]

(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1 - x^2\right)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$nI_{n+2} = 2^{n-1}3^{-\frac{1}{2}n} + (n-1)I_n.$$ [5]

(c) Find the exact value of I_5 giving the answer in the form $k\sqrt{3}$, where k is a rational number to be determined. [3]

7 It is given that $x = t^3 y$ and

$$t^3 \frac{d^2y}{dt^2} + (4t^3 + 6t^2)\frac{dy}{dt} + (13t^3 + 12t^2 + 6t)y = 61e^{\frac{1}{2}t}.$$

(a) Show that

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 61e^{\frac{1}{2}t}.$$ [4]

(b) Find the general solution for y in terms of t. [7]

8 **(a)** Find the values of a for which the system of equations

$$\begin{aligned} 3x + y + z &= 0, \\ ax + 6y - z &= 0, \\ ay - 2z &= 0, \end{aligned}$$

does not have a unique solution. [3]

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The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & -2 \end{pmatrix}.$$

(b) Use the characteristic equation of $\mathbf{A}$ to find the inverse of $\mathbf{A}^2$. [4]

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(c) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [7]